THE CLASS OF RECURSIVE FUNCTIONS

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In this note, we find the position of the predicate "α is recursive" in the Kleene arithmetical hierarchy.¹ The proof involves a use of topology; the key step is the application of Baire's category theorem to a suitable function space.

We use the notation of [1]. In addition, we write $(Ux)$ for the quantifier "there exist infinitely many x."

We designate the set of natural numbers by $\mathbb{N}$ and the class of mappings of $\mathbb{N}$ into $\mathbb{N}$ by $\mathbb{N}^\mathbb{N}$. We consider $\mathbb{N}$ as a topological space with the discrete topology, and $\mathbb{N}^\mathbb{N}$ as a topological space with the product topology. We review some known facts about $\mathbb{N}^\mathbb{N}$.

I. Provided with a suitable metric, $\mathbb{N}^\mathbb{N}$ is a complete metric space. (For it is the product of a countable number of discrete spaces.)

II. The nonrecursive functions are dense in $\mathbb{N}^\mathbb{N}$. (For a nonrecursive function remains nonrecursive if its values at a finite number of arguments are changed.)

III. If $R(\alpha)$ is a recursive predicate, then $\tilde{\alpha}(R(\alpha))$ is open and closed. (Since the complement of $\tilde{\alpha}(R(\alpha))$ is $\tilde{\alpha}(\overline{R(\alpha)})$, it is sufficient to prove $\tilde{\alpha}(R(\alpha))$ open. This follows from the fact that if $R(\alpha)$, then $R(\beta)$ for any function $\beta$ agreeing with $\alpha$ on a certain finite set of arguments.)

**Lemma.²** For every predicate $R(\alpha, x, y, z)$ there is a predicate $S(\alpha, x, y)$ recursive in $R$ such that

$$(x)(Ey)(z) R(\alpha, x, y, z) = (Ux)(y)S(\alpha, x, y)$$

for all $\alpha$.

**Proof.** Define

$$C(x, \alpha) = \gamma(Ez)R(\alpha, x, y, z),$$
$$D(x, \alpha) = \nu(y)\nu<y(y \in C(x, \alpha)), $$
$$E(v, \alpha) = \bigcup_{z<v}D(x, \alpha).$$

Then

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² This lemma was suggested to the author by a proof due to Hartley Rogers.
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\[ E(v, \alpha) \text{ is finite } \equiv (x)_{x < v} D(x, \alpha) \text{ is finite}\]
\[ \equiv (x)_{x < v} (Ey)(y \in C(x, \alpha))\]
\[ \equiv (x)_{x < v} (Ey)(z) R(\alpha, x, y, z).\]

Hence
\[ (x)(Ey)(z) R(\alpha, x, y, z) \equiv (Uv)(x)_{x < v} (Ey)(z) R(\alpha, x, y, z)\]
\[ \equiv (Uv) E(v, \alpha) \text{ is finite}.\] (1)

Now \( E(v, \alpha) \) is clearly recursively enumerable in \( R \) uniformly in \( v \) and \( \alpha \). Hence by the enumeration theorem, there is an \( e \) such that
\[ z \in E(v, \alpha) \equiv (Ew) T^R,\alpha(e, v, z, w).\]

We recall that for each \( v, z \) there is at most one \( w \) such that \( T^R,\alpha(e, v, z, w) \). It readily follows that
\[ (Uv) E(v, \alpha) \text{ is finite } \equiv (Ux)(y)(T^R,\alpha(e, (x)_0, (x)_1, (x)_2) \& lh(x)\]
\[ \leq 3 \& ((y)_0 > (x)_1 \rightarrow T^R,\alpha(e, (x)_0, (y)_0, (y)_1))).\]

Combining with (1), we get the lemma.

**Theorem.** The predicate “\( \alpha \) is recursive” can be written in the form
\( \exists x \exists y \exists z R(\alpha, x, y, z) \) with \( R \) recursive but cannot be written in the form \( \exists x (Ey)(z) R(\alpha, x, y, z) \) with \( R \) recursive.

**Proof.** The first part is immediate, since
\[ \alpha \text{ is recursive } \equiv (Ex)(y)(Ez) R(\alpha, x, y, z) \& U(z) = \alpha(y).\]

Now suppose “\( \alpha \) is recursive” could be written in the form \( \exists x (Ey)(z) R(\alpha, x, y, z) \) with \( R \) recursive. Then by the lemma and the enumeration theorem we would have
\[ \alpha \text{ is recursive } \equiv (Ux)(y)(\bar{T}_1(\bar{\alpha}(y), e, x))\] (2)

for a suitable \( e \). We shall show that this leads to a contradiction.

Let \( A(\alpha) = \bar{x}(y) \bar{T}(\bar{\alpha}(y), e, x) \). If \( \alpha \) is nonrecursive, then \( A(\alpha) \) is finite by (2). Since there are only countably many recursive functions and only countably many finite sets of natural numbers, there are only countably many different sets \( A(\alpha) \). Hence the sets \( \delta(A(\alpha) = B) \) with \( B \) a set of natural numbers form a countable covering of \( \mathbb{N}^N \). By I above and Baire’s category theorem, it follows that there is a set \( B \) such that, setting \( \mathcal{C} = \delta(A(\alpha) = B) \), \( \mathcal{C} \) has an interior point.

Since the nonrecursive functions are dense, there is a nonrecursive function \( \beta \) in \( \mathcal{C} \). Now for any \( x \),
\[ \delta(x \in A(\alpha)) = \delta(y)T_1^I(\alpha(y), e, x) = \bigcap_y \delta T_1^I(\alpha(y), e, x) \]
is closed by III above. Thus
\[ x \in B \rightarrow x \subseteq \delta(x \in A(\alpha)) \rightarrow \overline{\delta} = \delta(x \in A(\alpha)) \rightarrow x \in A(\beta). \]
Thus \( B \subseteq A(\beta) \). Since \( A(\beta) \) is finite, \( B \) is finite.

Let \( k = \max B \). Since \( \overline{\delta} \) has an interior point, there is a sequence number \( s > 0 \) such that
\[ (3) \quad (E y)(\alpha(y) = s) \rightarrow \alpha \in \overline{\delta}. \]
Set
\[ \gamma(0) = s, \]
\[ \gamma(n + 1) = \mu w(\text{Ext}(w, \gamma(n)) \& w > \gamma(n) \& T_1^I(w, e, k + n + 1)) \]
(where \( \text{Ext}(w, z) \) means that \( w \) and \( z \) are sequence numbers and that \( (w)_i = (z)_i \) for all \( i < lh(z) \)). We show that \( \gamma \) is well defined. Assume \( \gamma(n) \) is defined. Then \( \gamma(n) \) is a sequence number; so \( \gamma(n) = \delta(y) \) for some \( \delta \) and \( y \). By (3), \( \delta \in \overline{\delta} \). Hence there is a \( \sigma \) in \( \overline{\delta} \) such that \( \delta(y) = \delta(y) = \gamma(n) \). Now \( k + n + 1 \in B = A(\sigma) \). Hence for some \( u \),
\[ T_1^I(\delta(u), e, k + n + 1); \]
and we may suppose \( u > y \). If \( w = \delta(u) \), then \( \text{Ext}(w, \gamma(n)) \& w > \gamma(n) \& T_1^I(w, e, k + n + 1) \). Hence \( \gamma(n+1) \) is defined.

Clearly \( \gamma \) is recursive. Let \( \alpha(n) = (\gamma(n))_n^{-1} \); then \( \alpha \) is recursive. Now for each \( n \), \( \gamma(n+1) = \alpha(y) \) for some \( y \); so \( T_1^I(\alpha(y), e, k + n + 1) \) and hence \( k + n + 1 \in A(\alpha) \). It follows that \( A(\alpha) \) is finite. But this contradicts (2).

References


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