ON A CLASS OF UNIVALENT, STAR SHAPED MAPPINGS

ALBERT SCHILD\(^1\)

1. Introduction. Among all functions \( w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) regular and univalent in the unit circle, two classes of functions have been discussed extensively: The class of functions mapping the unit circle onto star shaped regions, characterized by \( \text{Re}\{zf''(z)/f'(z)\} \geq 0 \) for \( |z| < 1 \), and the class of functions mapping the unit circle onto convex regions, characterized by \( \text{Re}\{zf''(z)/f'(z)\} + 1 \geq 0 \) for \( |z| < 1 \).

This short paper will examine some of the geometric and analytic properties of a class of functions \( w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), which map the unit circle onto a region whose geometric nature is somewhat intermediate between star shaped and convex. The functions under consideration are to satisfy \( \text{Re}\{zf''(z)/f'(z)\} \geq 1/2 \) for all \( |z| < 1 \).

Interest in functions of this type can be traced back to two papers by A. Marx \([4]\) and E. Strohacker \([8]\) who have shown that for any function \( w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), which maps the unit circle onto a convex region, we have \( \text{Re}\{zf''(z)/f'(z)\} \geq 1/2 \), and as the function \( f(z) = z/(1+z) \) shows, the constant 1/2 cannot be improved. It is also clear that the converse is not true, i.e. functions for which \( \text{Re}\{zf''(z)/f'(z)\} \geq 1/2 \) need not map the unit circle onto a convex region. An example of this type is given by the function \( w = f(z) = z - 1/3 z^2 \) for which \( \text{Re}\{zf''(z)/f'(z)\} \geq 1/2 \), \( |z| \leq 1 \), yet the image region is not convex.

Recently, interest in this class of functions was roused again in a paper by R. F. Gabriel \([2]\). There it is shown that if \( p(z) \) is analytic and single valued for \( |z| < 1 \), and if we denote by \( w_1(z) = 1 + \sum_{n=2}^{\infty} a_n z^n \) and by \( w_2(z) = z + \sum_{n=2}^{\infty} b_n z^n \) two linearly independent solutions of \( w'' + p(z)w = 0 \), then \( f(z) = w_1(z)/w_2(z) = 1/z + \cdots \) will map \( |z| \leq 1 \) onto the exterior of a convex region if and only if \( \text{Re}\{zf''(z)/f'(z)\} \geq 1/2 \) for \( |z| < 1 \).

M. S. Robertson \([6]\) considered also the class of functions \( w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) for which \( \text{Re}\{zf''(z)/f'(z)\} \geq \alpha > 0, |z| < 1 \).

Notation. On the following pages we shall denote the class of functions \( w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) regular and univalent in the unit circle by \( S \).

The subclass of \( S \) which map \( |z| < 1 \) onto a star shaped region, i.e. for which \( \text{Re}\{zf''(z)/f'(z)\} \geq 0 \) by St.

Presented to the Society, January 28, 1958; received by the editors March 7, 1958 and, in revised form, April 21, 1958.

\(^1\) This work was aided by a grant from the Committee on Research and Publications of Temple University.

751
The subclass of $S_t$ which map $|z| < 1$ onto a “special” star shaped
region, i.e. for which \( \text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \geq 1/2 \) by $S_{t^*}$.

The subclass of $S$ which map $|z| < 1$ onto a convex region, i.e. for
which \( \text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} + 1 \geq 0 \) by $C$.

By [4] and [8] we obviously have:

1.1

\[ S \supset S_{t^*} \supset C. \]

2. Methods for constructing functions of class $S_{t^*}$. The interme-
diacy of $S_{t^*}$ between $S_t$ and $C$ suggests, that one might obtain func-
tions of class $S_{t^*}$ by “relaxing” the conditions on functions of class $C$
or by “strengthening” conditions on functions of class $S_t$. Both meth-
ods yield results.

**Theorem 2.1.** If $w = f(z) \in C$, then $g(z) = z(f'(z))^\alpha \in S_{t^*}$ for $0 < \alpha \leq 1/2$. The converse is true for $\alpha = 1/2$.

**Proof.** \( \text{Re}\left\{ \frac{zg'(z)}{g(z)} \right\} = \text{Re}\left\{ 1 + \alpha \frac{zf''(z)}{f'(z)} \right\} \geq 1 - \alpha \geq 1/2 \). For
the converse we have: \( \text{Re}\left\{ \frac{zf''(z)}{f'(z)} \right\} + 1 = \text{Re}\left\{ 2zf'(z)/g(z) \right\} - 1 \geq 0 \).
This should be compared with the well known result that if $f(z) \in C$
then $g(z) = zf'(z) \in S_t$ and conversely.

**Corollary 2.2.** Let \( |b_i| = 1 \) and \( \sum_{i=1}^{n} \mu_i \leq 1, \mu_i \geq 0 \), then $w = g(z)$
\[ z \prod_{i=1}^{n} (1 - b_i z)^{-\mu_i} \in S_{t^*}. \]

**Proof.** \( f(z) = \int_0^\infty \prod_{i=1}^{n} (1 - b_i z)^{-v_i} dt \in C \) if \( v_i \geq 0 \) and \( \sum_{i=1}^{n} v_i \leq 2, \)
\( |b_i| = 1 \). The corollary follows if we let $\alpha = 1/2$ in Theorem 2.1.

**Theorem 2.3.** If $w = f(z) \in S_t$, then $g(z) = z\left\{\frac{f'(z)}{f(z)}\right\}^\alpha \in S_{t^*}$ for
\[ 0 < \alpha \leq 1/2. \] The converse is true for $\alpha = 1/2$.

**Proof.** \( \text{Re}\left\{ \frac{zg'(z)}{g(z)} \right\} = \text{Re}\left\{ \alpha zf'(z)/f(z) + 1 - \alpha \right\} \geq 1 - \alpha \geq 1/2. \)
For the converse we have: \( \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} = \text{Re}\left\{ 2zf'(z)/g(z) - 1 \right\} \geq 0 \). (Marx [4] has the above theorem for $\alpha = 1/2$.)

The method of Theorem 2.3 for constructing functions of class
$S_{t^*}$ gives some insight into the geometrical nature of the region onto
which the unit circle is mapped by functions of class $S_{t^*}$. If $z = re^{i\theta}$ is
mapped by $w = f(z) \in S_t$ into $\text{Re} e^{i\psi}$, then a comparison of $r$ and $R$ and
$\theta$ and $\phi$ tells us about the amount of distortion. We notice that
$g(z) = \left\{\frac{zf(z)}{r} \right\}^{1/2} \in S_{t^*}$ reduces the amount of distortion effected by
$f(z) \in S_t$ by an “averaging” process, i.e. $z = re^{i\theta}$ will be mapped by $g(z)$
into $pe^{i\psi}$ where $p = (rR)^{1/2}$ and $\psi = (\theta + \phi)/2$.

3. Distortion theorems for functions of class $S_{t^*}$.

**Theorem 3.1.** For all $g(z) \in S_{t^*}$ we have $|z|/(1 + |z|) \leq |g(z)|$,$\leq|z|/(1 - |z|)$.
Proof. By Theorem 2.3 ($\alpha=1/2$) and the “Verzerrungs Satz” we have: $|z|/(1+|z|)^2 \leq \left| g(z)/z \right| \leq |z|/(1-|z|)^2$ and hence the Theorem follows. These inequalities are sharp for $g(z) = z/(1+z) \in \text{St}^*$, $z = \pm r$.

Theorem 3.2. For all $g(z) \in \text{St}^*$ the domain of values of $z g'(z)/g(z)$ is the circle with center at $1/(1-|z|^2)$ and radius $|z|/(1-|z|^2)$.

Proof. Let

$$G(z) = z g'(z)/g(z) - 1/2$$

and

$$H(z) = (2G(z) - 1)/(2G(z) + 1) = \left| z g'(z)/g(z) - 1 \right|/\left| z g'(z)/g(z) \right|.$$ 

Then $H(z)$ is regular for $|z| < 1$, $H(0) = 0$ and $|H(z)| < 1$ for $|z| < 1$. Hence the Lemma of Schwarz can be applied and we have for $|z| < 1$

$$\left| \frac{z g'(z)}{g(z)} - 1 \right| < |z| \quad \text{or} \quad \left| \frac{z g'(z)}{g(z)} \right| - 1 < |z| \left| \frac{z g'(z)}{g(z)} \right|.$$ 

But the domain defined by this inequality is the interior of the Circle of Apollonius with the line segment from $1/(1+|z|)$ to $1/(1-|z|)$ as a diameter, i.e. the interior of the circle with radius $|z|/(1-|z|^2)$ and center at $1/(1-|z|^2)$. The function $f(z) = z/(1+z) \in \text{St}^*$ shows that the theorem cannot be improved.

4. Some coefficient relations. It is well known that if $f(z) \in \text{St}$ then $|a_n| \leq n$. For functions of class $\text{St}^*$ we have:

Theorem 4.1. If $f(z) = z + \sum_{n=1}^{\infty} a_n z^n \in \text{St}^*$, then $|a_n| \leq 1$.

Proof. Let $p(z) = 2z f'(z)/f(z) - 1 = 1 + c_1 z + c_2 z^2 + \cdots$. Since $p(z)$ is regular and $\Re\{p(z)\} > 0$ for $|z| < 1$, therefore, by a well known lemma we have $|c_n| \leq 2$ for $n = 1, 2, 3, \cdots$.

Comparing coefficients we obtain $2(n-1)a_n = c_{n-1} + a_2 c_{n-2} + \cdots + a_{n-1} a_2$, and hence: $|a_n| \leq 1/(n-1) \{1 + |a_2| + \cdots + |a_{n-1}| \}$. It follows now by induction that $|a_n| \leq 1$ for $n = 1, 2, 3, \cdots$.

The function $f(z) = z/(1+z) = \sum_{n=1}^{\infty} z^n \in \text{St}^*$ shows that these inequalities are sharp.

For some kind of a converse we have:

Theorem 4.2. If $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and if $\sum_{n=2}^{\infty} (2n-1)|a_n| \leq 1$ then $g(z) \in \text{St}^*$.

Proof. The proof is based on a method used by A. Goodman [3]. We have:
\[
\frac{zg'(z)}{g(z)} - \frac{1}{2} = \frac{1 + 3a_2z + 5a_3z^2 + \cdots + (2n - 1)a_nz^{n-1} + \cdots}{2(1 + a_2z + a_3z^2 + \cdots + a_nz^{n-1} + \cdots)} = \frac{1}{2} + \sum_{n=1}^{\infty} b_nz^n
\]

where

\[ b_1 = a_2, \]
\[ b_2 = 2a_3 - a_2b_1, \]
\[ b_3 = 3a_4 - a_2b_2 - a_3b_1, \]
\[ \vdots \]
\[ b_{n-1} = (n - 1)a_n - a_2b_{n-2} - a_3b_{n-3} - \cdots - a_{n-1}b_1, \]

and therefore for \( n \geq 2 \)

\[
\sum_{k=1}^{n-1} b_k = \sum_{k=2}^{n} (k - 1)a_k - a_2\sum_{k=1}^{n-2} b_k - a_3\sum_{k=1}^{n-3} b_k - \cdots - a_n b_1. \quad (4.5)
\]

The inequality of the theorem implies that \( |b_1| = |a_2| \leq 1/3 \). It will now be shown by mathematical induction that for all \( n \) we have \( |\sum_{k=1}^{n} b_k| \leq 1/2 \). Assume that \( |\sum_{k=1}^{m} b_k| \leq 1/2 \) for \( m = 1, 2, 3, \cdots, n-2 \). Then (4.5) yields

\[
\left| \sum_{k=1}^{n-1} b_k \right| \leq \sum_{k=2}^{n} (k - 1)|a_k| + \frac{1}{2} \sum_{k=2}^{n-1} |a_k| \leq \sum_{k=2}^{n} (k - 1/2)|a_k| \leq 1/2 
\]

by the inequality of the theorem, and hence \( |\sum_{k=1}^{n} b_k| \leq 1/2 \) for all \( n \). From (4.3) it follows now that

\[
|zg'(z)/g(z) - 1| \leq 1/2 \text{ for } z = r. \quad (4.7)
\]

But the inequality of Theorem 4.2 and the special starshapedness of the image domain are invariant under rotations of the \( z \) and \( w \) planes. Hence any point in the unit circle may be placed in the interval \([0, 1]\), and thus (4.7) is valid throughout the unit circle i.e.

\[ \text{Re} \left\{ zg'(z)/g(z) \right\} \geq 1/2 \text{ for } |z| < 1. \]

**Corollary 4.8.** Let \( g(z) = z - \sum_{n=2}^{\infty} a_nz^n \), where all \( a_i \geq 0 \), then \( g(z) \in \text{St}^* \), if and only if \( \sum_{n=2}^{\infty} (2n-1)a_n \leq 1. \)
Proof. The sufficiency of the condition follows from the Theorem. For the necessity we have:

\[ \text{Re} \left\{ \frac{z g'(z)}{g(z)} - \frac{1}{2} \right\} = \text{Re} \left\{ \frac{1 - \sum_{n=2}^{\infty} (2n - 1)a_n z^{n-1}}{2 \left( 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right)} \right\}. \]

If \( \sum_{n=2}^{\infty} (2n - 1)a_n > 1 \), then we could find a positive value of \( z, r_0 \), for which the numerator of (4.9) would be negative and the denominator positive and hence \( \text{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} < 1/2 \) for that value of \( z \) and all positive values of \( z > r_0 \). (See also [7].)

5. The radius of special star shapedness. We define the radius of special star shapedness, \( r^* \), as the upper bound of the radii, \( r \), of circles \( |z| \leq r \), which are mapped by any function \( f(z) \in S \) onto a region of special star shapedness, i.e. that for all functions \( f(z) \in S \) we have \( \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1/2 \) for all \( |z| \leq r \). From the introduction it follows that \( r^* \) is larger than the bound of convexity (Rundungsschranke) and less than the bound of starlikeness. Therefore

\[ 2 - 3^{1/2} = .268 \cdots \leq r^* \leq .65 \cdots = \tanh \pi/4. \]

We have:

**Theorem 5.2.** \( r^* = 1/3 \) for all \( f(z) \in St \), and hence \( r^* \leq 1/3 \).

**Proof.** By [5] we have for any \( f(z) \in St \)

\[ \frac{1 - |z|}{1 + |z|} \leq \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \leq \frac{1 + |z|}{1 - |z|}. \]

Therefore \( \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq (1 - |z|)/(1 + |z|) \geq 1/2 \) for all \( |z| \leq 1/3 \). This result is sharp for \( f(z) = z(1+z)^{-2} \in St \) and \( z = 1/3 \).

**Theorem 5.3.** A lower bound for \( r^* \) is: \( r^* > .301 \cdots \).

**Proof.** If we let

\[ g(z) = f^{-1/2}(z^{-2}) = z + b_1/z + b_3/z^3 + \cdots + b_{2n-1}/z^{2n-1} + \cdots \]

then by Bieberbach's Flächensatz [1] we have

\[ \sum_{n=1}^{\infty} (2n - 1) |b_{2n-1}|^2 \leq 1. \]

The condition \( \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq 1/2 \) is equivalent to
(5.6) \[ \left| \frac{zf'(z)}{f(z)} \right| \leq \left| \frac{zf'(z)}{f(z)} - 1 \right| \]

i.e. by (5.4)

\[ \left| \frac{z^{-1/2}g'(z^{-1/2})}{g(z^{-1/2})} \right| \leq \left| \frac{z^{-1/2}g'(z^{-1/2})}{g(z^{-1/2})} - 1 \right| \]

or:

\[ \left| \frac{1 - b_1 z - 3b_3 z^2 - \cdots - (2n - 1)b_{2n-1} z^n}{1 + b_1 z + b_3 z^2 + \cdots + b_{2n-1} z^n} \right| \leq \left| \frac{1 - b_1 z - 3b_3 z^2 - \cdots - (2n - 1)b_{2n-1} z^n + \cdots}{1 + b_1 z + b_3 z^2 + \cdots + b_{2n-1} z^n + \cdots} - 1 \right| \]

(5.7)

or:

\[ \left| 1 - b_1 z - 3b_3 z^2 - \cdots - (2n - 1)b_{2n-1} z^n \right| \geq 2 \left| b_1 z + 2b_3 z^2 + 3b_5 z^3 + \cdots + nb_{2n-1} z^n + \cdots \right| . \]

A sufficient condition for this inequality to be satisfied is that

\[ 1 - \left| b_1 \right| r - 3 \left| b_3 \right| r^2 - \cdots - (2n - 1) \left| b_{2n-1} \right| r^n + \cdots \]

\[ \geq 2 \left| b_1 \right| r + 4 \left| b_3 \right| r^2 + \cdots \]

or:

\[ 3 \left| b_1 \right| r + 7 \left| b_3 \right| r^2 + 11 \left| b_5 \right| r^3 + \cdots \]

\[ + (4n - 1) \left| b_{2n-1} \right| r^n + \cdots \leq 1 \]

or

\[ 2 \left\{ \left| b_1 \right| r + 3 \left| b_3 \right| r^2 + \cdots + (2n - 1) \left| b_{2n-1} \right| r^n + \cdots \right\} \]

\[ + \left\{ \left| b_1 \right| r + \left| b_3 \right| r^2 + \cdots + \left| b_{2n-1} \right| r^n + \cdots \right\} \leq 1 . \]

(5.8)

If in the first parenthesis of inequality (5.8) welet \( c_n = (2n - 1)^{1/2} \left| b_{2n-1} \right| \)

and \( d_n = (2n - 1)^{1/2} \cdot r^n \) then the Inequality of Schwarz

\[ \sum c_n \cdot d_n \leq \left( \sum c_n^2 \right)^{1/2} \cdot \left( \sum d_n^2 \right)^{1/2} \]

gives

\[ \left| b_1 \right| r + 3 \left| b_3 \right| r^2 + \cdots + (2n - 1) \left| b_{2n-1} \right| r^n + \cdots \]

(5.9)

\[ \leq \left( \left| b_1 \right|^2 + 3 \left| b_3 \right|^2 + \cdots + (2n - 1) \left| b_{2n-1} \right|^2 + \cdots \right)^{1/2} \]

\[ \cdot (r^2 + \cdots + (2n - 1) r^{2n} + \cdots)^{1/2} \]

and therefore by (5.5)
Similarly, if in the second parenthesis of inequality (5.8) we let
c_n=(2n-1)^{1/2}|b_{2n-1}|, d_n=r^n(2n-1)^{-1/2}
and apply the Inequality of Schwarz again, we get:
\[
\left| b_1 \right| r + \left| b_3 \right| r^2 + \cdots + \left| b_{2n-1} \right| r^n + \cdots \\
\leq \left( \left| b_1 \right|^2 + 3 \left| b_3 \right|^2 + \cdots + (2n-1) \left| b_{2n-1} \right| + \cdots \right)^{1/2} \\
\cdot \left( r^2 + \frac{r^4}{3} + \cdots + \frac{r^{2n}}{2n-1} + \cdots \right)^{1/2}
\]
and again by (5.5)
\[
\left| b_1 \right| r + \left| b_3 \right| r^2 + \cdots + \left| b_{2n-1} \right| r^n + \cdots \\
\leq \left( r^2 + \frac{r^4}{3} + \cdots + \frac{r^{2n}}{2n-1} + \cdots \right)^{1/2} = \left( \frac{r}{2} \ln \frac{1+r}{1-r} \right)^{1/2}.
\]
Substituting (5.10) and (5.12) into (5.8) we have:
\[
\frac{2r(1 + r^2)^{1/2}}{1 - r^2} + \left( \frac{r}{2} \log \frac{1+r}{1-r} \right)^{1/2} < 1.
\]
This will be satisfied for \( |z| = r < .301 \cdots \), i.e. \( r^* > .301 \cdots \).

**Bibliography**