ON A CLASS OF UNIVERSAL ORDERED SETS

ELLIOTT MENDELSON

An ordered set $B$ is said to be $\aleph_\alpha$-universal if and only if every ordered set of power $\aleph_\alpha$ is similar to a subset of $B$. Let $U_{\omega_\alpha}$ be the lexicographically ordered set of all sequences of 0's and 1's of type $\omega_\alpha$; and let $H_\alpha$ be the subset of $U_{\omega_\alpha}$ consisting of all sequences \( \{x_\xi\}_{\xi<\omega_\alpha} \) for which there is some $\xi_0<\omega_\alpha$ such that $x_{\xi_0}=1$ and, for $\xi>\xi_0$, $x_\xi=0$.

$H_0$, being countable, dense, and without first or last element, is similar to the set of rationals in their natural order, and therefore, is $\aleph_0$-universal. Sierpiński [2] has shown (as a direct consequence of his theorem that $H_{\alpha+1}$ is an $\eta_{\alpha+1}$-set) that, for any $\alpha$, $H_{\alpha+1}$ is $\aleph_{\alpha+1}$-universal. Gillman [1] has given a demonstration that, for any limit ordinal $\alpha$, $H_\alpha$ is $\aleph_\alpha$-universal. The purpose of this note is to give a very simple proof of these results, which does not depend on the ordinal $\alpha$.

**Theorem.** $H_\alpha$ is $\aleph_\alpha$-universal.

**Proof.** Let $A$ be an ordered set of power $\aleph_\alpha$. Fix some well-ordering $\{a_\beta\}_{\beta<\omega_\alpha}$ of $A$. Let $<$ denote the order in $A$. Define a function $\phi$ from $A$ into $H_\alpha$ in the following way. Let $a_\tau$ be an element of $A$, and $\beta<\omega_\alpha$. Then the $\beta$th component $\phi_\beta(a_\tau)$ of $\phi(a_\tau)$ is defined by:

\[
\phi_\beta(a_\tau) = \begin{cases} 
1 & \text{if } \beta \leq \tau \text{ and } a_\beta \leq a_\tau, \\
0 & \text{otherwise}.
\end{cases}
\]

Now, let $a_\tau$ and $a_\sigma$ be any elements of $A$, with $a_\tau < a_\sigma$. Clearly, if $\beta \leq \sigma$, $\phi_\beta(a_\sigma) = \phi_\beta(a_\tau)$. But, $\phi_\sigma(a_\sigma) = 1$ and $\phi_\sigma(a_\tau) = 0$. Hence, $\phi(a_\tau)$ pre-
cedes \( \phi(a_\alpha) \) in \( H_\alpha \). Thus, \( \phi \) is a one-one order-preserving mapping of \( A \) into \( H_\alpha \). Q.E.D.

Note that \( H_0 \) is \( \aleph_0 \)-universal and of power \( \aleph_0 \). Since \( \aleph_{\alpha+1} = 2^{\aleph_\alpha} \), \( \aleph_{\alpha+1} \) is of power \( \aleph_\alpha \) if and only if \( 2^{\aleph_\alpha} = \aleph_{\alpha+1} \). Finally, for limit ordinals \( \alpha \), \( \aleph_\alpha = \sum_{\beta < \alpha} 2^{\aleph_\beta} \), and, therefore, \( H_\alpha \) is of power \( \aleph_\alpha \) if and only if, for every \( \beta < \alpha \), \( 2^{\aleph_\beta} \leq \aleph_\alpha \) (and, hence, if \( 2^{\aleph_\beta} = \aleph_{\beta+1} \) for all \( \beta < \alpha \)). For \( \alpha > 0 \), it seems to be an open problem to prove, without additional cardinality assumptions, the existence of an \( \aleph_\alpha \)-universal ordered set of power \( \aleph_\alpha \).

**Bibliography**


**Harvard University**