THE EQUIVALENCE OF TWO EXTENSIONS OF
LEBESGUE AREA

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The area of a triangle in a Banach space was defined in [5] and then Lebesgue's definition for the area of a continuous function (surface) was applicable to a surface in a Banach space. If $X$ is a continuous function on a closed simply-connected Jordan region $J$ into a Banach space $B$ let $L_B(X)$ denote its area in accordance with this definition. Let $m$ be the space of bounded sequences [1]. It was shown that isometric functions$^1$ on $J$ into $m$ have the same $L_m$ area. If $X$ is continuous on $J$ into a metric space $M$ there is a continuous function $x$ on $J$ into $m$ which is isometric with $X$. Define $L(X)$ to be equal to $L_m(x)$.

If $X$ is continuous on $J$ into a Euclidean space $E$, then $L_E(X)$ is the classical Lebesgue area of $X$. It was shown that $L(X) = L_E(X)$ and hence it seemed reasonable to call $L$ an extension of Lebesgue area which applied to surfaces in a metric space.

Now let us consider those functions which are continuous on $J$ into a Banach space $B$. There are two Lebesgue type areas available, $L$ and $L_B$. Morrey's representation theorem [3] and the cyclic additivity theory were used to show that $L = L_B$ under a variety of conditions, in particular, as mentioned above, if $B = E$.

The hypothesis in Morrey's theorem that the functions range in $E$ was too strong a restriction to enable us to conclude that $L(X) = L_B(X)$ for all Banach spaces $B$ and continuous functions $X$ on $J$ into $B$. In [6] Morrey's theorem is extended to apply to surfaces in a metric space. We shall see that this version of Morrey's theorem can be used to show that $L$ and $L_B$ are equivalent, whenever $L_B$ is applicable.

Let $J$ be contained in the $u, v$ plane. According to Cesari [2], a function $X$ on $J$ into $E$ is a $D$-mapping if each component of $X$ is A.C.T. in $J^0$, the interior of $J$, and if all of the partial derivatives are square summable over $J^0$. This definition can be generalized so as to apply to functions with range in a Banach space [6]. The property of being a $D$-mapping is invariant under an isometric transformation. If $X$ is a $D$-mapping in $B$ then $X$ is of class $L_B$ in the sense of [5] (see [4, IV. 4.33]).

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$^1$ If $X$ and $Y$ map $J$ into metric spaces $M$ and $N$ respectively, then $X$ and $Y$ are isometric if $\text{dist}_M(X(u), X(v)) = \text{dist}_N(Y(u), Y(v))$ for all $u, v \in J$. 

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The equivalence of $L$ and $L_B$, whenever the latter is defined, is a result of the cyclic additivity theory, the fact that Fréchet equivalent functions have the same Lebesgue area, $L$ or $L_B$, and Theorems 1 and 2 (from [5] and [6], respectively).

**THEOREM 1.** If range $X \subset B$ then $L(X) \leq L_B(X)$. If $X$ is of class $L_B$ then $L(X) = L_B(X)$.

**THEOREM 2.** If range $x \subset m$, if $x$ is light, and if $L(x) < +\infty$, then there is a $D$-mapping $y$ which is Fréchet equivalent to $x$.

**THEOREM 3.** If $X$ is continuous on $J$ into $B$ then $L(X) = L_B(X)$.

**Proof.** We can use Theorem 1 and the cyclic additivity theory to assume, without loss of generality, that $X$ is light and that $L(X) < +\infty$. Take $x$ isometric with $X$, range $x \subset m$. According to Theorem 2 there is a $D$-mapping $y$ which is Fréchet equivalent to $x$. For each $u \in J$ choose $v$ to satisfy $x(v) = y(u)$ and define $Y(u) = X(v)$. (If $v_1$ and $v_2$ correspond to $u$ then $x(v_1) = x(v_2)$ which implies that $X(v_1) = X(v_2)$.) It is easy to see that $X$ and $Y$ are Fréchet equivalent and that $X$ and $y$ are isometric. Thus $Y$ is a $D$-mapping and Theorem 1 enables us to conclude that $L_B(X) = L_B(Y) = L(Y) = L(X)$.

**References**


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