ON THE STRUCTURE OF MAXIMUM MODULUS ALGEBRAS

WALTER RUDIN

Let \( U, K, \) and \( C \) denote the open unit disc, the closed unit disc, and the unit circumference, respectively. In \([1]\), an algebra \( \mathfrak{A} \) of continuous complex functions defined on \( K \) was said to be a maximum modulus algebra on \( K \) if for every \( f \in \mathfrak{A} \)
\[
\max_{z \in K} |f(z)| = \max_{z \in C} |f(z)|,
\]
i.e., if every \( f \in \mathfrak{A} \) attains its maximum modulus on \( C \). As a matter of convenience, we shall, in this paper, abbreviate “maximum modulus algebra on \( K \)” to “M-algebra.”

Examples of M-algebras which come to mind immediately are (a) the algebra \( \mathfrak{A} \) consisting of all functions which are continuous on \( K \) and analytic in \( U \), (b) any subalgebra of \( \mathfrak{A} \), (c) any algebra \( \mathfrak{B} \) which is equivalent to an algebra of analytic functions via a homeomorphism of \( K \), i.e., any algebra \( \mathfrak{B} \) with which there is associated a homeomorphism \( h \) of \( K \) into the complex plane, such that every \( f \in \mathfrak{B} \) is of the form
\[
f(z) = f^*(h(z)) \quad (z \in K)
\]
for some \( f^* \) which is continuous on \( h(K) \) and analytic in \( h(U) \).

Does this list contain all M-algebras? The results of \([1]\) seemed to point toward a positive answer. In fact, the main theorem of \([1]\), stated in somewhat different form, is as follows:

**Theorem 1.** Consider the following two conditions which an M-algebra \( \mathfrak{B} \) may satisfy:

(i) There is a function \( h \in \mathfrak{B} \) which is a homeomorphism of \( K \).
(ii) \( \mathfrak{B} \) contains a nonconstant function \( \phi \in \mathfrak{A} \).

Condition (i) alone implies that \( \mathfrak{B} \) is equivalent to an algebra of analytic functions, via \( h \). Conditions (i) and (ii) together imply that \( \mathfrak{B} \subset \mathfrak{A} \).

The next question that arises naturally is whether (ii) alone implies that \( \mathfrak{B} \subset \mathfrak{A} \). That this is not so was shown by an example in \([3]\); there an M-algebra \( \mathfrak{A}' \) was constructed which was generated by two functions \( f \) and \( g \), where \( f \) was analytic (and not constant) in \( U \) and \( g \)

Presented to the Society February 22, 1958; received by the editors March 5, 1958.

1 The author is a Research Fellow of the Alfred P. Sloan Foundation.

708
was not analytic. It is clear, incidentally, that $\mathfrak{R}'$ cannot be equivalent to any algebra of analytic functions; for if $f(z) = f^*(h(z))$ and $g(z) = g^*(h(z))$, as in (2), with $f^*$ and $g^*$ analytic, then the analyticity of $f$ implies that $h$ is analytic (compare [1, p. 452]), and this forces $g$ to be analytic.

The algebra $\mathfrak{R}'$ does not separate points on $K$ (i.e., there exist $z_1 \in K$, $z_2 \in K$ such that $z_1 \neq z_2$ but $\phi(z_1) = \phi(z_2)$ for every $\phi \in \mathfrak{R}'$). Thus the question arises whether the conclusion "$\mathfrak{R} \subset \mathfrak{A}$" of Theorem 1 can be rescued if we assume (ii) and some weakened form of (i), for instance, if we replace (i) by the requirement that $\mathfrak{A}$ should separate points on $K$ (so that there is a canonical homeomorphism of $K$ into the maximal ideal space of the Banach algebra $\mathfrak{A}$, the uniform closure of $\mathfrak{A}$; we may assume without loss of generality that $\mathfrak{A}$ contains the constants [1, p. 450]). The answer, given in the present paper, settles the question raised in [2], and is again negative:

**Theorem 2.** There exists a finitely generated $M$-algebra $\mathfrak{R}$ such that

- (a) $\mathfrak{R}$ separates points on $K$,
- (b) $\mathfrak{R}$ contains nonconstant functions which are analytic in $U$, and
- (c) $\mathfrak{R}$ contains functions which are not analytic in $U$.

**Proof.** Let $P$ be a perfect, totally disconnected, bounded subset of the plane, whose two-dimensional Lebesgue measure is positive. Let $Q$ be the set of all points $(w_1, w_2, w_3, w_4)$ in the space of 4 complex variables (i.e., the 8-dimensional euclidean space $E^8$) such that $w_i \in P$ for $i = 1, 2, 3, 4$; $Q$ is the cartesian product $P \times P \times P \times P$, embedded in $E^8$ in a natural way. Note that both $P$ and $Q$ are homeomorphic to the Cantor set.

There exists a simple closed curve $J$ in the plane such that $P \subset J$. Let $D$ be the interior of $J$. The crux of the proof will be the construction of 4 complex continuous functions $h_1, \ldots, h_4$ on $K$, with the following properties:

(a) There exists a subset $H$ of $C$, homeomorphic to the Cantor set, such that the mapping

$$z \mapsto (h_1(z), h_2(z), h_3(z), h_4(z))$$

is one-to-one on $H$ and maps $H$ onto $Q$.

(\beta) $h_1 \in A$ and $h_1(K-H) \subset D$;

(\gamma) The set $\{h_2, h_3, h_4\}$ separates points on $K-H$;

(\delta) There is an arc $L \subset U$ on which $h_2$ is constant.

(We note that (\delta) could be replaced by practically any condition which assures nonanalyticity.)

Once we have these functions, we can prove the theorem quite
rapidly. Since $P$ has positive measure, there exist nonconstant complex functions $q_1$, $q_2$, $q_3$ which are continuous in the plane, analytic in the complement of $P$ (including the point at infinity), such that the set $\{q_1, q_2, q_3\}$ separates points in the plane; for the proof of this, see [4, pp. 826–827]. Let $\mathfrak{A}$ be the algebra generated by the functions $f_{ij}$, where

\begin{equation}
 f_{ij}(z) = q_i(h_j(z)) \quad (i = 1, 2, 3; j = 1, 2, 3, 4; z \in K).
\end{equation}

Condition (β) implies that $f_{ii} \in A$; condition (δ) implies that $f_{ii} \in A$; conditions (α), (β), (γ) together imply that the set $\{h_1, h_2, h_3, h_4\}$ separates points on $K$, and hence $\mathfrak{A}$ separates points on $K$. There only remains the verification that $\mathfrak{A}$ is an $M$-algebra.

Every member of $\mathfrak{A}$ is of the form

\begin{equation}
 f(z) = g(f_{ij}(z)) = g(q_i(h_j(z))),
\end{equation}

where $g$ is a polynomial in 12 variables. Put

\begin{equation}
 \phi(w_1, w_2, w_3, w_4) = g(q_i(w_j)).
\end{equation}

If we keep $w_2, w_3, w_4$ fixed, then $\phi$, as a function of $w_1$, is analytic in the complement of $P$. The maximum modulus theorem therefore implies that there is a point $w_1^* \in P$ such that

\begin{equation}
 |\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2, w_3, w_4)|.
\end{equation}

Keeping $w_1^*, w_3, w_4$ fixed, and then repeating this procedure twice more, we find that there is a point $(w_1^*, w_2^*, w_3^*, w_4^*) \in Q$ such that

\begin{equation}
 |\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2^*, w_3^*, w_4^*)|
\end{equation}

for all $(w_1, w_2, w_3, w_4)$. By (α) there is a point $z^* \in H$ such that $h_j(z^*) = w_j^*$ $(j = 1, \ldots, 4)$, and a glance at (4), (5), and (7) shows that

\begin{equation}
 |f(z)| \leq |f(z^*)|
\end{equation}

for all $z \in K$.

Thus $\mathfrak{A}$ is an $M$-algebra, and Theorem 2 follows.

We now turn to the construction of the functions $h_1, \ldots, h_4$ and of the set $\mathcal{H}$.

Let $E$ be a perfect subset of $C$, of measure zero. There exist complex continuous functions $\phi_1, \ldots, \phi_4$, defined on $E$, such that the mapping

\begin{equation}
 t \rightarrow (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))
\end{equation}

is a homeomorphism of $E$ onto $Q$. By the theorem proved in [3], there exists a function $f_1 \in A$, such that $f_1(t) = \phi_1(t)$ for all $t \in E$ and
such that \( f(K) \subset D \cup J \). Let \( K' \) be the closed convex hull of \( E \), let \( \psi \) be a conformal map of \( K \) onto \( K' \) (i.e., \( \psi \) is a homeomorphism of \( K \) onto \( K' \) which is conformal in the interior of \( K \)), and put \( H = \psi^{-1}(E) \).

Define

\[ h_1(z) = f_1(\psi(z)) \quad (z \in K), \]

and

\[ h_j(z) = \phi_j(\psi(z)) \quad (j = 2, 3, 4; z \in H). \]

Then condition (\( \beta \)) holds, and if we can extend \( h_2, h_3, h_4 \) from \( H \) to \( K \) so that (\( \gamma \)) and (\( \delta \)) are satisfied, the proof will be complete, since (\( \alpha \)) is implied by our choice of \( \{ \phi_j \} \).

Triangulate \( K - H \); each compact subset of \( K - H \) will be covered by a finite collection of triangles (some of these will be curvilinear), and every point of \( H \) will be a limit point of the set \( T \) of vertices. Pick two vertices \( t', t'' \in U \) which are joined by an edge of our triangulation, and define \( h_j(t) \) for \( j = 2, 3, 4 \) and \( t \in T \) such that \( h_j \) is continuous on \( H \cup T \), such that

\[ h_2(t') = h_2(t'') = 0, \]

and such that the points \( h(t) = (h_2(t), h_3(t), h_4(t)) \) are in general position in \( E^6 \); i.e., no \( m + 2 \) of these points lie in any linear \( m \)-space, for \( m = 1, \ldots, 4 \).

Let \( \Delta \) be one of our triangles, with vertices \( t_1, t_2, t_3 \). Define \( h_j(z) \) for \( z \in \Delta \) so that the mapping

\[ z \rightarrow (h_2(z), h_3(z), h_4(z)) \]

is a homeomorphism of \( \Delta \) onto the (rectilinear) triangle whose vertices are the points \( h(t_1), h(t_2), h(t_3) \) in \( E^6 \).

The functions \( h_j \) are now extended to \( K \) and are continuous on \( K \).

Since the points \( h(t) \) are in general position, no two triangles whose vertices are among these points will intersect, except possibly in a common vertex or a common edge. It follows that condition (\( \gamma \)) is satisfied; and (12) shows that condition (\( \delta \)) also holds, with the interval \([t', t'']\) for \( L \).

This completes the proof of the theorem. It seems quite likely that another proof can be given by exhibiting an example with fewer generators; their number can perhaps be pushed down to 2, but different methods are needed for this.

In conclusion, we pose another problem:

Suppose \( \mathcal{A} \) is an \( M \)-algebra such that \( \mathcal{A} \cap \mathcal{R} \) separates points on \( K \). Does it follow that \( \mathcal{A} \subset \mathcal{R} \)?
ON A CLASS OF UNIVERSAL ORDERED SETS

ELLIOTT MENDELSON

An ordered set \( B \) is said to be \( \aleph_\alpha \)-universal if and only if every ordered set of power \( \aleph_\alpha \) is similar to a subset of \( B \). Let \( U_{\omega_\alpha} \) be the lexicographically ordered set of all sequences of 0's and 1's of type \( \omega_\alpha \); and let \( H_\alpha \) be the subset of \( U_{\omega_\alpha} \) consisting of all sequences \( \{x_\xi\}_{\xi<\omega_\alpha} \) for which there is some \( \xi_0<\omega_\alpha \) such that \( x_{\xi_0}=1 \) and, for \( \xi>\xi_0, x_\xi=0 \).

\( H_0 \), being countable, dense, and without first or last element, is similar to the set of rationals in their natural order, and therefore, is \( \aleph_0 \)-universal. Sierpiński [2] has shown (as a direct consequence of his theorem that \( H_{\alpha+1} \) is an \( \eta_\alpha+1 \)-set) that, for any \( \alpha \), \( H_{\alpha+1} \) is \( \aleph_{\alpha+1} \)-universal. Gillman [1] has given a demonstration that, for any limit ordinal \( \alpha \), \( H_\alpha \) is \( \aleph_\alpha \)-universal. The purpose of this note is to give a very simple proof of these results, which does not depend on the ordinal \( \alpha \).

**Theorem.** \( H_\alpha \) is \( \aleph_\alpha \)-universal.

**Proof.** Let \( A \) be an ordered set of power \( \aleph_\alpha \). Fix some well-ordering \( \{a_\beta\}_{\beta<\omega_\alpha} \) of \( A \). Let \( < \) denote the order in \( A \). Define a function \( \phi \) from \( A \) into \( H_\alpha \) in the following way. Let \( a_\tau \) be an element of \( A \), and \( \beta<\omega_\alpha \). Then the \( \beta \)th component \( \phi_\beta(a_\tau) \) of \( \phi(a_\tau) \) is defined by:

\[
\phi_\beta(a_\tau) = \begin{cases} 
1 & \text{if } \beta \leq \tau \text{ and } a_\beta \leq a_\tau, \\
0 & \text{otherwise}.
\end{cases}
\]

Now, let \( a_\tau \) and \( a_\sigma \) be any elements of \( A \), with \( a_\tau < a_\sigma \). Clearly, if \( \beta \leq \sigma \), \( \phi_\beta(a_\sigma) \geq \phi_\beta(a_\tau) \). But, \( \phi_\sigma(a_\sigma) = 1 \) and \( \phi_\sigma(a_\tau) = 0 \). Hence, \( \phi(a_\tau) \) pre-