

ON THE STRUCTURE OF MAXIMUM MODULUS ALGEBRAS

WALTER RUDIN¹

Let U , K , and C denote the open unit disc, the closed unit disc, and the unit circumference, respectively. In [1], an algebra \mathcal{R} of continuous complex functions defined on K was said to be a *maximum modulus algebra on K* if for every $f \in \mathcal{R}$

$$(1) \quad \max_{z \in K} |f(z)| = \max_{z \in C} |f(z)|,$$

i.e., if every $f \in \mathcal{R}$ attains its maximum modulus on C . As a matter of convenience, we shall, in this paper, abbreviate "maximum modulus algebra on K " to " M -algebra."

Examples of M -algebras which come to mind immediately are (a) the algebra \mathcal{A} consisting of all functions which are continuous on K and analytic in U , (b) any subalgebra of \mathcal{A} , (c) any algebra \mathcal{B} which is equivalent to an algebra of analytic functions via a homeomorphism of K , i.e., any algebra \mathcal{B} with which there is associated a homeomorphism h of K into the complex plane, such that every $f \in \mathcal{B}$ is of the form

$$(2) \quad f(z) = f^*(h(z)) \quad (z \in K)$$

for some f^* which is continuous on $h(K)$ and analytic in $h(U)$.

Does this list contain all M -algebras? The results of [1] seemed to point toward a positive answer. In fact, the main theorem of [1], stated in somewhat different form, is as follows:

THEOREM 1. *Consider the following two conditions which an M -algebra \mathcal{R} may satisfy:*

- (i) *There is a function $h \in \mathcal{R}$ which is a homeomorphism of K .*
- (ii) *\mathcal{R} contains a nonconstant function $\phi \in \mathcal{R}$.*

Condition (i) alone implies that \mathcal{R} is equivalent to an algebra of analytic functions, via h . Conditions (i) and (ii) together imply that $\mathcal{R} \subset \mathcal{A}$.

The next question that arises naturally is whether (ii) alone implies that $\mathcal{R} \subset \mathcal{A}$. That this is not so was shown by an example in [3]; there an M -algebra \mathcal{R}' was constructed which was generated by two functions f and g , where f was analytic (and not constant) in U and g

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¹ The author is a Research Fellow of the Alfred P. Sloan Foundation.

was not analytic. It is clear, incidentally, that \mathcal{R}' cannot be equivalent to any algebra of analytic functions; for if $f(z) = f^*(h(z))$ and $g(z) = g^*(h(z))$, as in (2), with f^* and g^* analytic, then the analyticity of f implies that h is analytic (compare [1, p. 452]), and this forces g to be analytic.

The algebra \mathcal{R}' does not separate points on K (i.e., there exist $z_1 \in K, z_2 \in K$ such that $z_1 \neq z_2$ but $\phi(z_1) = \phi(z_2)$ for every $\phi \in \mathcal{R}'$). Thus the question arises whether the conclusion " $\mathcal{R} \subset \mathcal{A}$ " of Theorem 1 can be rescued if we assume (ii) and some weakened form of (i), for instance, if we replace (i) by the requirement that \mathcal{R} should separate points on K (so that there is a canonical homeomorphism of K into the maximal ideal space of the Banach algebra $\overline{\mathcal{R}}$, the uniform closure of \mathcal{R} ; we may assume without loss of generality that \mathcal{R} contains the constants [1, p. 450]). The answer, given in the present paper, settles the question raised in [2], and is again negative:

THEOREM 2. *There exists a finitely generated M -algebra \mathcal{R} such that*

- (a) \mathcal{R} separates points on K ,
- (b) \mathcal{R} contains nonconstant functions which are analytic in U , and
- (c) \mathcal{R} contains functions which are not analytic in U .

PROOF. Let P be a perfect, totally disconnected, bounded subset of the plane, whose two-dimensional Lebesgue measure is positive. Let Q be the set of all points (w_1, w_2, w_3, w_4) in the space of 4 complex variables (i.e., the 8-dimensional euclidean space E^8) such that $w_i \in P$ for $i = 1, 2, 3, 4$; Q is the cartesian product $P \times P \times P \times P$, embedded in E^8 in a natural way. Note that both P and Q are homeomorphic to the Cantor set.

There exists a simple closed curve J in the plane such that $P \subset J$. Let D be the interior of J . The crux of the proof will be the construction of 4 complex continuous functions h_1, \dots, h_4 on K , with the following properties:

(α) *There exists a subset H of C , homeomorphic to the Cantor set, such that the mapping*

$$z \rightarrow (h_1(z), h_2(z), h_3(z), h_4(z))$$

is one-to-one on H and maps H onto Q .

(β) $h_1 \in \mathcal{A}$ and $h_1(K - H) \subset D$;

(γ) The set $\{h_2, h_3, h_4\}$ separates points on $K - H$;

(δ) There is an arc $L \subset U$ on which h_2 is constant.

(We note that (δ) could be replaced by practically any condition which assures nonanalyticity.)

Once we have these functions, we can prove the theorem quite

rapidly. Since P has positive measure, there exist nonconstant complex functions q_1, q_2, q_3 which are continuous in the plane, analytic in the complement of P (including the point at infinity), such that the set $\{q_1, q_2, q_3\}$ separates points in the plane; for the proof of this, see [4, pp. 826–827]. Let \mathfrak{R} be the algebra generated by the functions f_{ij} , where

$$(3) \quad f_{ij}(z) = q_i(h_j(z)) \quad (i = 1, 2, 3; j = 1, 2, 3, 4; z \in K).$$

Condition (β) implies that $f_{i1} \in A$; condition (δ) implies that $f_{i2} \notin A$; conditions (α) , (β) , (γ) together imply that the set $\{h_1, h_2, h_3, h_4\}$ separates points on K , and hence \mathfrak{R} separates points on K . There only remains the verification that \mathfrak{R} is an M -algebra.

Every member of \mathfrak{R} is of the form

$$(4) \quad f(z) = g(f_{ij}(z)) = g(q_i(h_j(z))),$$

where g is a polynomial in 12 variables. Put

$$(5) \quad \phi(w_1, w_2, w_3, w_4) = g(q_i(w_j)).$$

If we keep w_2, w_3, w_4 fixed, then ϕ , as a function of w_1 , is analytic in the complement of P . The maximum modulus theorem therefore implies that there is a point $w_1^* \in P$ such that

$$(6) \quad |\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2, w_3, w_4)|.$$

Keeping w_1^*, w_3, w_4 fixed, and then repeating this procedure twice more, we find that there is a point $(w_1^*, w_2^*, w_3^*, w_4^*) \in Q$ such that

$$(7) \quad |\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2^*, w_3^*, w_4^*)|$$

for all (w_1, w_2, w_3, w_4) . By (α) there is a point $z^* \in H$ such that $h_j(z^*) = w_j^*$ ($j=1, \dots, 4$), and a glance at (4), (5), and (7) shows that

$$(8) \quad |f(z)| \leq |f(z^*)|$$

for all $z \in K$.

Thus \mathfrak{R} is an M -algebra, and Theorem 2 follows.

We now turn to the construction of the functions h_1, \dots, h_4 and of the set H .

Let E be a perfect subset of C , of measure zero. There exist complex continuous functions ϕ_1, \dots, ϕ_4 , defined on E , such that the mapping

$$(9) \quad t \rightarrow (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$$

is a homeomorphism of E onto Q . By the theorem proved in [3], there exists a function $f_1 \in A$, such that $f_1(t) = \phi_1(t)$ for all $t \in E$ and

such that $f(K) \subset D \cup J$. Let K' be the closed convex hull of E , let ψ be a conformal map of K onto K' (i.e., ψ is a homeomorphism of K onto K' which is conformal in the interior of K), and put $H = \psi^{-1}(E)$. Define

$$(10) \quad h_1(z) = f_1(\psi(z)) \quad (z \in K),$$

and

$$(11) \quad h_j(z) = \phi_j(\psi(z)) \quad (j = 2, 3, 4; z \in H).$$

Then condition (β) holds, and if we can extend h_2, h_3, h_4 from H to K so that (γ) and (δ) are satisfied, the proof will be complete, since (α) is implied by our choice of $\{\phi_j\}$.

Triangulate $K - H$; each compact subset of $K - H$ will be covered by a finite collection of triangles (some of these will be curvilinear), and every point of H will be a limit point of the set T of vertices. Pick two vertices $t', t'' \in U$ which are joined by an edge of our triangulation, and define $h_j(t)$ for $j = 2, 3, 4$ and $t \in T$ such that h_j is continuous on $H \cup T$, such that

$$(12) \quad h_2(t') = h_2(t'') = 0,$$

and such that the points $h(t) = (h_2(t), h_3(t), h_4(t))$ are in general position in E^6 ; i.e., no $m+2$ of these points lie in any linear m -space, for $m = 1, \dots, 4$.

Let Δ be one of our triangles, with vertices t_1, t_2, t_3 . Define $h_j(z)$ for $z \in \Delta$ so that the mapping

$$(13) \quad z \rightarrow (h_2(z), h_3(z), h_4(z))$$

is a homeomorphism of Δ onto the (rectilinear) triangle whose vertices are the points $h(t_1), h(t_2), h(t_3)$ in E^6 .

The functions h_j are now extended to K and are continuous on K .

Since the points $h(t)$ are in general position, no two triangles whose vertices are among these points will intersect, except possibly in a common vertex or a common edge. It follows that condition (γ) is satisfied; and (12) shows that condition (δ) also holds, with the interval $[t', t'']$ for L .

This completes the proof of the theorem. It seems quite likely that another proof can be given by exhibiting an example with fewer generators; their number can perhaps be pushed down to 2, but different methods are needed for this.

In conclusion, we pose another problem:

Suppose \mathcal{R} is an M -algebra such that $\mathcal{R} \cap \mathcal{R}$ separates points on K . Does it follow that $\mathcal{R} \subset \mathcal{Q}$?

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UNIVERSITY OF ROCHESTER

ON A CLASS OF UNIVERSAL ORDERED SETS

ELLIOTT MENDELSON

An ordered set B is said to be \aleph_α -universal if and only if every ordered set of power \aleph_α is similar to a subset of B . Let U_{ω_α} be the lexicographically ordered set of all sequences of 0's and 1's of type ω_α ; and let H_α be the subset of U_{ω_α} consisting of all sequences $\{x_\xi\}_{\xi < \omega_\alpha}$ for which there is some $\xi_0 < \omega_\alpha$ such that $x_{\xi_0} = 1$ and, for $\xi > \xi_0$, $x_\xi = 0$.

H_0 , being countable, dense, and without first or last element, is similar to the set of rationals in their natural order, and therefore, is \aleph_0 -universal. Sierpiński [2] has shown (as a direct consequence of his theorem that $H_{\alpha+1}$ is an $\eta_{\alpha+1}$ -set) that, for any α , $H_{\alpha+1}$ is $\aleph_{\alpha+1}$ -universal. Gillman [1] has given a demonstration that, for any limit ordinal α , H_α is \aleph_α -universal. The purpose of this note is to give a very simple proof of these results, which does not depend on the ordinal α .

THEOREM. H_α is \aleph_α -universal.

PROOF. Let A be an ordered set of power \aleph_α . Fix some well-ordering $\{a_\beta\}_{\beta < \omega_\alpha}$ of A . Let $<$ denote the order in A . Define a function ϕ from A into H_α in the following way. Let a_τ be an element of A , and $\beta < \omega_\alpha$. Then the β th component $\phi_\beta(a_\tau)$ of $\phi(a_\tau)$ is defined by:

$$\phi_\beta(a_\tau) = \begin{cases} 1 & \text{if } \beta \leq \tau \text{ and } a_\beta \leq a_\tau, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let a_τ and a_σ be any elements of A , with $a_\tau < a_\sigma$. Clearly, if $\beta \leq \sigma$, $\phi_\beta(a_\sigma) \geq \phi_\beta(a_\tau)$. But, $\phi_\sigma(a_\sigma) = 1$ and $\phi_\sigma(a_\tau) = 0$. Hence, $\phi(a_\tau)$ pre-