ON THE STRUCTURE OF MAXIMUM MODULUS ALGEBRAS

WALTER RUDIN

Let \( U, K, \) and \( C \) denote the open unit disc, the closed unit disc, and the unit circumference, respectively. In \([1]\), an algebra \( \mathfrak{A} \) of continuous complex functions defined on \( K \) was said to be a maximum modulus algebra on \( K \) if for every \( f \in \mathfrak{A} \)

\[
\max_{z \in K} |f(z)| = \max_{z \in C} |f(z)|,
\]

i.e., if every \( f \in \mathfrak{A} \) attains its maximum modulus on \( C \). As a matter of convenience, we shall, in this paper, abbreviate “maximum modulus algebra on \( K \)” to “M-algebra.”

Examples of \( M \)-algebras which come to mind immediately are (a) the algebra \( \mathfrak{A} \) consisting of all functions which are continuous on \( K \) and analytic in \( U \), (b) any subalgebra of \( \mathfrak{A} \), (c) any algebra \( \mathfrak{B} \) which is equivalent to an algebra of analytic functions via a homeomorphism of \( K \), i.e., any algebra \( \mathfrak{B} \) with which there is associated a homeomorphism \( h \) of \( K \) into the complex plane, such that every \( f \in \mathfrak{B} \) is of the form

\[
f(z) = f^*(h(z)) \quad (z \in K)
\]

for some \( f^* \) which is continuous on \( h(K) \) and analytic in \( h(U) \).

Does this list contain all \( M \)-algebras? The results of \([1]\) seemed to point toward a positive answer. In fact, the main theorem of \([1]\), stated in somewhat different form, is as follows:

**Theorem 1.** Consider the following two conditions which an \( M \)-algebra \( \mathfrak{A} \) may satisfy:

(i) There is a function \( h \in \mathfrak{A} \) which is a homeomorphism of \( K \).

(ii) \( \mathfrak{A} \) contains a nonconstant function \( \phi \in \mathfrak{A} \).

Condition (i) alone implies that \( \mathfrak{A} \) is equivalent to an algebra of analytic functions, via \( h \). Conditions (i) and (ii) together imply that \( \mathfrak{A} \subset \mathfrak{B} \).

The next question that arises naturally is whether (ii) alone implies that \( \mathfrak{A} \subset \mathfrak{B} \). That this is not so was shown by an example in \([3]\); there an \( M \)-algebra \( \mathfrak{A}' \) was constructed which was generated by two functions \( f \) and \( g \), where \( f \) was analytic (and not constant) in \( U \) and \( g \)

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was not analytic. It is clear, incidentally, that \( \mathcal{R}' \) cannot be equivalent to any algebra of analytic functions; for if \( f(z) = f^*(h(z)) \) and \( g(z) = g^*(h(z)) \), as in (2), with \( f^* \) and \( g^* \) analytic, then the analyticity of \( f \) implies that \( h \) is analytic (compare [1, p. 452]), and this forces \( g \) to be analytic.

The algebra \( \mathcal{R}' \) does not separate points on \( K \) (i.e., there exist \( z_1 \in K, z_2 \in K \) such that \( z_1 \neq z_2 \) but \( \phi(z_1) = \phi(z_2) \) for every \( \phi \in \mathcal{R}' \). Thus the question arises whether the conclusion "\( \mathcal{R} \subseteq \mathcal{A} \)" of Theorem 1 can be rescued if we assume (ii) and some weakened form of (i), for instance, if we replace (i) by the requirement that \( \mathcal{R} \) should separate points on \( K \) (so that there is a canonical homeomorphism of \( K \) into the maximal ideal space of the Banach algebra \( \mathcal{R} \), the uniform closure of \( \mathcal{R} \); we may assume without loss of generality that \( \mathcal{R} \) contains the constants [1, p. 450]). The answer, given in the present paper, settles the question raised in [2], and is again negative:

**Theorem 2.** There exists a finitely generated \( M \)-algebra \( \mathcal{A} \) such that

(a) \( \mathcal{A} \) separates points on \( K \),
(b) \( \mathcal{A} \) contains nonconstant functions which are analytic in \( U \), and
(c) \( \mathcal{A} \) contains functions which are not analytic in \( U \).

**Proof.** Let \( P \) be a perfect, totally disconnected, bounded subset of the plane, whose two-dimensional Lebesgue measure is positive. Let \( Q \) be the set of all points \((w_1, w_2, w_3, w_4)\) in the space of 4 complex variables (i.e., the 8-dimensional euclidean space \( E^8 \)) such that \( w_i \in P \) for \( i = 1, 2, 3, 4 \); \( Q \) is the cartesian product \( P \times P \times P \times P \), embedded in \( E^8 \) in a natural way. Note that both \( P \) and \( Q \) are homeomorphic to the Cantor set.

There exists a simple closed curve \( J \) in the plane such that \( P \subseteq J \). Let \( D \) be the interior of \( J \). The crux of the proof will be the construction of 4 complex continuous functions \( h_1, \ldots, h_4 \) on \( K \), with the following properties:

(\( \alpha \)) There exists a subset \( H \) of \( C \), homeomorphic to the Cantor set, such that the mapping

\[
z \rightarrow (h_1(z), h_2(z), h_3(z), h_4(z))
\]

is one-to-one on \( H \) and maps \( H \) onto \( Q \).

(\( \beta \)) \( h_1 \in A \) and \( h_1(K-H) \subseteq D \);

(\( \gamma \)) The set \( \{h_2, h_3, h_4\} \) separates points on \( K-H \);

(\( \delta \)) There is an arc \( L \subseteq U \) on which \( h_2 \) is constant.

(We note that (\( \delta \)) could be replaced by practically any condition which assures nonanalyticity.)

Once we have these functions, we can prove the theorem quite
rapidly. Since $P$ has positive measure, there exist nonconstant complex functions $q_1$, $q_2$, $q_3$ which are continuous in the plane, analytic in the complement of $P$ (including the point at infinity), such that the set $\{q_1, q_2, q_3\}$ separates points in the plane; for the proof of this, see [4, pp. 826–827]. Let $\mathfrak{A}$ be the algebra generated by the functions $f_{ij}$, where

$$f_{ij}(z) = q_i(h_j(z)) \quad (i = 1, 2, 3; j = 1, 2, 3, 4; z \in K).$$

Condition (β) implies that $f_{ij} \in A$; condition (δ) implies that $f_{ij} \in A$; conditions (α), (β), (γ) together imply that the set $\{h_1, h_2, h_3, h_4\}$ separates points on $K$, and hence $\mathfrak{A}$ separates points on $K$. There only remains the verification that $\mathfrak{A}$ is an $M$-algebra.

Every member of $\mathfrak{A}$ is of the form

$$f(z) = g(f_{ij}(z)) = g(q_i(h_j(z))),$$

where $g$ is a polynomial in 12 variables. Put

$$\phi(w_1, w_2, w_3, w_4) = g(q_i(w_j)).$$

If we keep $w_2, w_3, w_4$ fixed, then $\phi$, as a function of $w_1$, is analytic in the complement of $P$. The maximum modulus theorem therefore implies that there is a point $w_1^* \in P$ such that

$$|\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2, w_3, w_4)|.$$

Keeping $w_1^*, w_3, w_4$ fixed, and then repeating this procedure twice more, we find that there is a point $(w_1^*, w_2^*, w_3^*, w_4^*) \in Q$ such that

$$|\phi(w_1, w_2, w_3, w_4)| \leq |\phi(w_1^*, w_2^*, w_3^*, w_4^*)|$$

for all $(w_1, w_2, w_3, w_4)$. By (α) there is a point $z^* \in H$ such that $h_j(z^*) = w_j^*$ ($j = 1, \cdots, 4$), and a glance at (4), (5), and (7) shows that

$$|f(z)| \leq |f(z^*)|$$

for all $z \in K$.

Thus $\mathfrak{A}$ is an $M$-algebra, and Theorem 2 follows.

We now turn to the construction of the functions $h_1, \cdots, h_4$ and of the set $H$.

Let $E$ be a perfect subset of $C$, of measure zero. There exist complex continuous functions $\phi_1, \cdots, \phi_4$, defined on $E$, such that the mapping

$$t \mapsto (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$$

is a homeomorphism of $E$ onto $Q$. By the theorem proved in [3], there exists a function $f_1 \in A$, such that $f_1(t) = \phi_1(t)$ for all $t \in E$ and
such that \( f(K) \subset D \cup J \). Let \( K' \) be the closed convex hull of \( E \), let \( \psi \) be a conformal map of \( K \) onto \( K' \) (i.e., \( \psi \) is a homeomorphism of \( K \) onto \( K' \) which is conformal in the interior of \( K \)), and put \( H = \psi^{-1}(E) \). Define

\[
(10) \quad h_1(z) = f_1(\psi(z)) \quad (z \in K),
\]

and

\[
(11) \quad h_j(z) = \phi_j(\psi(z)) \quad (j = 2, 3, 4; z \in H).
\]

Then condition (\( \beta \)) holds, and if we can extend \( h_2, h_3, h_4 \) from \( H \) to \( K \) so that (\( \gamma \)) and (\( \delta \)) are satisfied, the proof will be complete, since (\( \alpha \)) is implied by our choice of \( \{ \phi_j \} \).

Triangulate \( K - H \); each compact subset of \( K - H \) will be covered by a finite collection of triangles (some of these will be curvilinear), and every point of \( H \) will be a limit point of the set \( T \) of vertices. Pick two vertices \( t', t'' \in U \) which are joined by an edge of our triangulation, and define \( h_j(t) \) for \( j = 2, 3, 4 \) and \( t \in T \) such that \( h_j \) is continuous on \( \bar{H} \cap T \), such that

\[
(12) \quad h_2(t') = h_2(t'') = 0,
\]

and such that the points \( h(t) = (h_2(t), h_3(t), h_4(t)) \) are in general position in \( E^6 \); i.e., no \( m + 2 \) of these points lie in any linear \( m \)-space, for \( m = 1, \ldots, 4 \).

Let \( \Delta \) be one of our triangles, with vertices \( t_1, t_2, t_3 \). Define \( h_j(z) \) for \( z \in \Delta \) so that the mapping

\[
(13) \quad z \to (h_2(z), h_3(z), h_4(z))
\]

is a homeomorphism of \( \Delta \) onto the (rectilinear) triangle whose vertices are the points \( h(t_1), h(t_2), h(t_3) \) in \( E^6 \).

The functions \( h_j \) are now extended to \( K \) and are continuous on \( K \).

Since the points \( h(t) \) are in general position, no two triangles whose vertices are among these points will intersect, except possibly in a common vertex or a common edge. It follows that condition (\( \gamma \)) is satisfied; and (12) shows that condition (\( \delta \)) also holds, with the interval \([t', t'']\) for \( L \).

This completes the proof of the theorem. It seems quite likely that another proof can be given by exhibiting an example with fewer generators; their number can perhaps be pushed down to 2, but different methods are needed for this.

In conclusion, we pose another problem:

Suppose \( \mathcal{A} \) is an \( M \)-algebra such that \( \mathcal{A} \cap \mathcal{A} \) separates points on \( K \). Does it follow that \( \mathcal{A} \subset \mathcal{A} \)?
ON A CLASS OF UNIVERSAL ORDERED SETS

ELLIOTT MENDELSON

An ordered set $B$ is said to be $\mathfrak{S}_\alpha$-universal if and only if every ordered set of power $\mathfrak{S}_\alpha$ is similar to a subset of $B$. Let $U_{\omega_\alpha}$ be the lexicographically ordered set of all sequences of 0's and 1's of type $\omega_\alpha$; and let $H_\alpha$ be the subset of $U_{\omega_\alpha}$ consisting of all sequences $\{x_\xi\}_{\xi<\omega_\alpha}$ for which there is some $\xi_0<\omega_\alpha$ such that $x_{\xi_0}=1$ and, for $\xi>\xi_0$, $x_\xi=0$.

$H_0$, being countable, dense, and without first or last element, is similar to the set of rationals in their natural order, and therefore, is $\mathfrak{S}_0$-universal. Sierpiński [2] has shown (as a direct consequence of his theorem that $H_{\alpha+1}$ is an $\eta_{\alpha+1}$-set) that, for any $\alpha$, $H_{\alpha+1}$ is $\mathfrak{S}_{\alpha+1}$-universal. Gillman [1] has given a demonstration that, for any limit ordinal $\alpha$, $H_\alpha$ is $\mathfrak{S}_\alpha$-universal. The purpose of this note is to give a very simple proof of these results, which does not depend on the ordinal $\alpha$.

THEOREM. $H_\alpha$ is $\mathfrak{S}_\alpha$-universal.

Proof. Let $A$ be an ordered set of power $\mathfrak{S}_\alpha$. Fix some well-ordering $\{a_\beta\}_{\beta<\omega_\alpha}$ of $A$. Let $<$ denote the order in $A$. Define a function $\phi$ from $A$ into $H_\alpha$ in the following way. Let $a_\tau$ be an element of $A$, and $\beta<\omega_\alpha$. Then the $\beta$th component $\phi_\beta(a_\tau)$ of $\phi(a_\tau)$ is defined by:

$$\phi_\beta(a_\tau) = \begin{cases} 1 & \text{if } \beta \leq \tau \text{ and } a_\beta \leq a_\tau, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $a_\tau$ and $a_\sigma$ be any elements of $A$, with $a_\tau < a_\sigma$. Clearly, if $\beta \leq \sigma$, $\phi_\beta(a_\tau) \geq \phi_\beta(a_\sigma)$. But, $\phi_\sigma(a_\sigma) = 1$ and $\phi_\sigma(a_\tau) = 0$. Hence, $\phi(a_\tau)$ pre-

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