ON THE ITERATION OF TRANSFORMATIONS IN
NONCOMPACT MINIMAL DYNAMICAL SYSTEMS

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Let $A$ be a Hausdorff space, $\phi$ a continuous mapping of $A$ into itself. It is the purpose of the present paper to discuss various topics centering around the following question: If $g$ is a bounded continuous function on $A$, does there exist a bounded continuous function $f$ on $A$ such that $f(\phi a) - f(a) = g(a)$ for all $a$ in $A$? Suppose that for each $a_0$ in $A$, the set $\{\phi^n a_0, n \geq 0\}$ is dense in $A$. Theorem 1 asserts that a necessary and sufficient condition for the existence of such an $f$ is that $\left| \sum_{k=0}^{j} g(\phi^k a) \right|$ should be uniformly bounded for all positive $j$ and all points $a$ of $A$. For homeomorphisms of compact spaces, this result was previously obtained by Gottschalk and Hedlund [5, Theorem 14.11, p. 135].

A related problem for linear operators in a Banach space is obtained by letting $X$ be the Banach space of bounded continuous functions on $A$ with the uniform norm, $T$ the linear transformation of $X$ into itself defined by $(Tf)(a) = f(\phi a)$, $a \in A$. In terms of $X$ and $T$, Theorem 1 states that $g$ will lie in the range of $(I - T)$ if and only if the sequence of norms $\left\| \sum_{k=0}^{j} T^k g \right\|$ is uniformly bounded for all positive $j$. In a reflexive Banach space, this characterization of the range of $(I - T)$ is valid for any linear transformation $T$ for which $\left\| T^n \right\|$ is bounded for all $n$. A sufficient condition in a general Banach space would seem to require an assumption that the elements $\{\sum_{k=0}^{j} T^k g\}$ lie for all $j$ in a fixed weakly compact subset $K$ of $X$. It would be interesting to obtain a proof of Theorem 1 along these lines. We shall content ourselves with showing by these methods that if $m$ is a totally-finite measure on a $\sigma$-algebra on $A$, $L^\infty(m)$ the space of $m$-essentially bounded measurable functions, $\phi$ a measure preserving mapping of $A$ into $A$, then in order that for an element $g$ in $L^\infty(m)$, there should exist an $f$ in $L^\infty(m)$ such that $f(\phi a) - f(a) = g(a)$ a.e. in $m$, it is necessary and sufficient that $m$-ess. sup. $\left| \sum_{k=0}^{j} g(\phi^k a) \right|$ should be uniformly bounded for all positive $j$.

Such a result raises another sort of question. For a topological space $A$ if $g$ is continuous and $f$ is a solution of the equation $f(\phi a)$
$-f(a) = g(a)$, with $f$ lying in some larger class of functions, must $f$ be necessarily continuous after change on some negligible set? We place the question in a more definite setting. Let $A_1$ be a Hausdorff space, $A_2$ a compact topological group, $\phi$ a homeomorphism of $A_1$ onto itself such that $A_1$ is a minimal orbit closure under $\phi$, $\psi_0$ a continuous map of $A_1$ into $A_2$. Suppose there exists a Baire function $h$ from $A_1$ to $A_2$ satisfying the relation

$$h(\phi a) = \psi_0(a_1) \cdot h(a_1),$$

for all $a_1$ outside some set of the first category in $A_1$. Then if $A_1$ is a Baire space, $h$ is continuous after change on a set of the first category.

A similar result is valid if $A_2$ is merely a compact space, $\psi_0$ a homeomorphism of $A_2$ onto itself which generates an equicontinuous transformation group of $A_2$, and (1) is replaced by

$$h(\phi a) = \psi_0(ha_1).$$

In this form, the result has been established by S. Kakutani in [7] by rather different methods. One interesting feature of the present proof is that it is valid also under the following hypotheses: $A_1$ a measure space with a measure $m$ such that all open sets are measurable and have positive measure, $\phi$ maps null sets on null sets, $h$ a function from $A_1$ to $A_2$ continuous on the complement of a set of zero measure. Then $h$ is continuous on the whole of $A_1$ after replacement on a set of measure zero. An extension is given for functional equations of a more general type than (1)'.

1: Let $A$ be a Hausdorff space, $\phi$ a continuous mapping of $A$ into itself. We assume that $A$ is a minimal orbit closure under $\phi$, i.e., for every $a_0$ in $A$, the closure of the set $\{\phi^n a_0, n \geq 0\}$ coincides with $A$. Let $B$ be another Hausdorff space, $\psi$ a continuous mapping of the Cartesian product $A \times B$ into $B$.

We define a continuous mapping $\pi$ of $A \times B$ into itself by setting $\pi(a, b) = (\phi a, \psi(a, b))$. If $\pi^n$ is the $n$th iterate of $\pi$, $O(a, b)$ is the orbit of $(a, b)$ under the mapping $\pi$, i.e. $O(a, b) = \bigcup_{n \geq 0} \{\pi^n(a, b)\}$, then we let $F(a, b)$ be the closure of $O(a, b)$ in $A \times B$. Let $p_A$ and $p_B$ be the projection mappings of $A \times B$ on its first and second components respectively, $p_A(a, b) = a$, $p_B(a, b) = b$. We shall assume in the following that for each point $(a, b)$ in $A \times B$, $p_B(F(a, b))$ is contained in a compact subset of $B$.

Consider the family $J$ of subsets of $A \times B$, where $J = \{F | F$ is a nonempty closed subset of $A \times B; (a, b) \in F$ implies that $\pi(a, b) \in F; p_B(F)$ is contained in a compact subset of $B}\}$. Since for any point
Lemma 1. If $F \in J$, then $p_A(F) = A$.

Proof. Let $(a_0, b_0)$ be a point of $F$. Since $\pi^n(a_0, b_0) \in F$, $p_A \pi^n(a_0, b_0) \in p_A(F)$. Thus $p_A(F)$ contains the dense set $\{ \phi^n a_0 \}$ and therefore is dense in $A$. On the other hand, $F$ is closed in $A \times B$, $F \subseteq A \times \text{Cl}(p_B(F))$, and $\text{Cl}(p_B(F))$ is compact in $B$. Therefore, $p_A(F)$ is closed in $A$ [1, Exercise 8, p. 68]. Since $p_A(F)$ is dense and closed in $A$, $p_A(F) = A$.

Lemma 2. $J$ has a minimal element under inclusion. Every orbit closure $F(a, b)$ contains a minimal element of $J$.

Proof. By the Lemma of Zorn, it suffices to prove that every subfamily of $J$ which is linearly ordered with respect to inclusion has a lower bound in $J$. Let $L = \{ F_\alpha \}$ be such a family. Then $F_0 = \bigcap_\alpha F_\alpha$ is a closed invariant set under $\pi$ while $p_B(F_0)$ is certainly contained in a compact subset of $B$. To prove that $F_0 \subseteq J$, we must show $F_0 \neq \emptyset$. Let $a_0$ be a point of $A$, $G_\alpha = F_\alpha \cap p_A^{-1}(a_0)$. By Lemma 1, $G_\alpha$ is a family of closed sets in $A \times B$ such that every finite subfamily has a nonempty intersection. Moreover, each $G_\alpha$ is a closed subset of $a_0 \times \text{Cl}(p_B(F_\alpha))$, which is compact since it is mapped homeomorphically by $p_B$ on the compact set $\text{Cl}(p_B(F_\alpha))$. Since all the $G_\alpha$ are compact, $G_0 = \bigcap_\alpha G_\alpha$ is nonempty, and, since $G_0 \subseteq F_0$, $F_0$ is nonempty.

Let $\xi$ be a homeomorphism of $B$ onto itself commuting with $\psi$, i.e. such that $\psi(a, \xi b) = \xi \psi(a, b)$ for all $a \in A$, $b \in B$. Let $S_\xi$ be the homeomorphism of $A \times B$ onto itself defined by $S_\xi(a, b) = (a, \xi b)$.

Lemma 3. Let $F_0$ be a minimal element of $J$ and suppose that for a fixed point $a$ in $A$, the points $(a, b)$ and $(a, b_1)$ lie in $F$. Suppose further that there exists a homeomorphism $\xi$ of $B$ onto $B$ commuting with $\psi$, such that $\xi b = b_1$. Then $S_\xi F_0 = F_0$.

Proof. From the fact that $\xi$ commutes with $\psi$, we see that $S_\xi \pi(a, b) = (\phi a, \xi \psi(a, b)) = (\phi a, \psi(a, \xi b)) = \pi S_\xi(a, b)$. Thus $S_\xi \pi^n = \pi^n S_\xi$, and $S_\xi (O(a, b)) = O(a, \xi b)$. Since $S_\xi$ is a homeomorphism, $S_\xi F(a, b) = F(a, \xi b)$. Since $F_0$ is a minimal element of $J$, $F_0 = F(a, b) = F(a, b_1)$. But $S_\xi F_0 = S_\xi F(a, b) = F(a, \xi b) = F(a, b_1) = F_0$.

Theorem 1. Let $\phi$ be a continuous mapping of the Hausdorff space $A$ into itself with $A$ a minimal orbit closure under $\phi$. Let $g$ be a bounded continuous function from $A$ to the $n$-dimensional Euclidean space $R^n$. In order that there should exist a bounded continuous function $f$ from $A$ to $R^n$ such that $f(\phi a) - f(a) = g(a)$ for all $a$ in $A$, it is necessary and
sufficient that there exist a constant \( M > 0 \) with

\[
(2) \quad \sup_{a \in A} \left| \sum_{k=0}^{j} g(\phi^k a) \right| \leq M \text{ for all } j \geq 0.
\]

**Proof of Theorem 1.** Necessity is obvious for if \( g(a) = f(\phi a) - f(a) \), then \( |\sum_{k=0}^{j} g(\phi^k a)| = |f(\phi^{j+1} a) - f(a)| \leq 2 \sup |f(a)| \).

To prove sufficiency, we specialize our preceding discussion by taking \( B = \mathbb{R}^n \) and setting \( \psi(a, r) = r + g(a) \) for \( a \in A, r \in \mathbb{R}^n \). The corresponding mapping \( \pi \) is defined by \( \pi(a, r) = (\phi a, r + g(a)) \). The condition (2) is equivalent to the fact that the orbit of any point \( (a, r) \) under \( \pi \) has a bounded and hence precompact image in \( \mathbb{R}^n \) under the projection map \( \rho_{\mathbb{R}^n} \) of \( A \times \mathbb{R}^n \) into \( \mathbb{R}^n \). Hence the conclusions of Lemmas 1, 2, and 3 are valid for this mapping \( \pi \). Let \( F_0 \) be a minimal closed invariant set in \( A \times \mathbb{R}^n \) with respect to \( \pi \). Suppose that for some point \( a \) in \( A \), \( \rho_{\mathbb{R}^n}^{-1}(a) \cap F_0 \) contained two distinct points \( (a, r), (a, r_1) \). Let \( \xi = r - r_1 \), the homeomorphism of \( \mathbb{R}^n \) onto itself defined by \( \xi(r) = r + \xi \). Then \( \xi \) commutes with \( \psi, \xi(r_1) = r \), and Lemma 3 is applicable. Thus if \( S_\xi(a, r) = (a, r + \xi), S_\xi F_0 = F_0 \). But then \( S_m^{\xi} F_0 = F_0 \) for any positive integer \( m \), contradicting the boundedness of the second component for elements of \( F_0 \). Thereby, we have shown that \( F_0 \) has at most one point \( (a, r) \) for a given \( a \in A \).

Let \( f \) be the function from \( A \) to \( \mathbb{R}^n \) defined uniquely by the condition \( (a, f(a)) \in F_0 \). By Lemma 1, \( f \) is defined on all of \( A \). \( f \) can be considered as a function from \( A \) to the compact set \( \text{Cl}(\rho_{\mathbb{R}^n} F_0) \). Since \( F_0 \), the graph of \( f \), is closed, \( f \) is continuous [1, Exercise 12, p. 68]. Since \( \pi(a, f(a)) \in F_0 \), we have \( (\phi a, f(a) + g(a)) \in F_0 \), i.e. \( f(\phi a) = f(a) + g(a) \).

**Remark.** Following a remark of Kakutani, we note that the existence of a minimal subset in \( J \) under the hypotheses of Theorem 1 can be proved in an elementary way without the use of Zorn’s Lemma or the Axiom of Choice. Let \( F_0 = F(a_0, r_0) \) for a fixed element \( (a_0, r_0) \) in \( A \times \mathbb{R}^n \). We shall show that \( F_0 \) is a minimal element of \( J \). It suffices to show that if \( (a, r) \in F_0 \), then \( (a_0, r_0) \in F(a, r) \). We note first that if \( (a_0, r_1) \in F_0 \), and \( \xi = r_1 - r_0 \), then \( S_\xi F_0 = S_\xi F(a_0, r_0) = F(a_0, r_1) \subseteq F_0 \). If \( \xi \neq 0 \), then \( S_\xi^m F_0 \subseteq F_0 \) for all \( m > 0 \), contradicting (2). Thus \( \xi = 0 \) and \( (a_0, r_0) \) is the only point in \( \rho_A^{-1}(a_0) \cap F_0 \). But \( \rho_A^{-1}(a_0) \cap F(a, r) \) is contained in \( \rho_A^{-1}(a_0) \cap F_0 \) and is nonempty by Lemma 1. It follows that \( (a_0, r_0) \in F(a, r) \) and \( F_0 \) is minimal.

2. Let \( X \) be a Banach space, \( T \) a continuous linear transformation of \( X \) into itself.

**Lemma 4.** A sufficient condition for \( g \) in \( X \) to lie in the range of
\[(I-T)\text{ is that the set of elements } \{ \sum_{k=0}^{j-1} T^k g \}\text{ should lie for } j \geq 0 \text{ in a fixed weakly compact subset } K \text{ of } X.\]

**Proof.** By theorems of Eberlein and M. Krein (cf. \([4]\)), the convex closure \(K'\) of \(K \cup \{0\}\) is weakly sequentially compact. By the principle of uniform boundedness, there exists \(M > 0\) such that \(\left\| \sum_{k=0}^{j-1} T^k g \right\| \leq M\) for \(j \geq 0\). Thus if we set \(g_n = g - n^{-1} \sum_{k=0}^{j-1} T^k g\), then \(g_n\) will converge strongly to \(g\) as \(n \to \infty\). Furthermore, each \(g_n\) lies in the range of \((I-T)\) since \(g_n = (I-T) \left\{ n^{-1} \sum_{k=0}^{j-1} (\sum_{k=0}^{j-1} T^k g) \right\}\). Let us set \(h_k = \sum_{k=0}^{j-1} T^k g, f_n = \{ \sum_{k=0}^{j-1} h_j \} \cdot n^{-1}\). Then \(g_n = (I-T)f_n\), while the \(f_n\) lie for all \(n\) in the weakly sequentially compact set \(K'\). Choose a subsequence \(f_{n_i}\) converging weakly to an element \(f\) of \(X\) as \(i \to \infty\). Then \((I-T)f_{n_i}\) converges weakly to \((I-T)f\). But \(g_{n_i} = (I-T)f_{n_i}\) converges strongly to \(g\). Hence \(g = (I-T)f\).

**Lemma 5.** Let \(X\) be a reflexive Banach space, \(T\) a continuous linear transformation of \(X\) into itself. A sufficient condition that \(g\) lie in the range of \((I-T)\) is that \(\left\| \sum_{k=0}^{j-1} T^k g \right\|\) be uniformly bounded for \(j \geq 0\). If \(\left\| T^n \right\| \leq M'\) for \(n \geq 0\), the condition is also necessary.

**Proof.** The necessity is obvious, since if \(g = (I-T)f\), \(\left\| \sum_{k=0}^{j-1} T^k g \right\| = \left\| f - T^{i+1}f \right\| \leq 2M'\). Sufficiency follows from Lemma 4 since every closed ball about zero in a reflexive space is weakly compact.

**Theorem 2.** Let \(A\) be a measure space with a totally finite measure \(m\), \(\phi\) a measure preserving mapping of \(A\) into \(A\). In order that for a function \(g\) in \(L^{\infty}(m)\), there should exist an \(f \in L^{\infty}(m)\) such that \(f(\phi a) - f(a) = g(a)\) a.e. in \(m\), it is necessary and sufficient that

\[
\text{m-ess. sup. } \left| \sum_{k=0}^{j} g(\phi^k a) \right|
\]

should be uniformly bounded for \(j \geq 0\).

**Proof of Theorem 2.** Choose a value of \(p, 1 < p < \infty\). Let \(T\) mapping \(L^p(m)\) into itself be defined by \((Tf)(a) = f(\phi a), a \in A\). Then \(\left\| Tf \right\|_{L^p} = \left\| f \right\|_{L^p}\), while \(\left\| f \right\|_{L^p} \leq m(A)^{1/p} \left\| f \right\|_{L^\infty}\) for \(f \in L^p \cap L^\infty\). Since necessity is obvious, we consider only sufficiency. Let \(g\) be our given function from \(L^\infty\). Since \(\left\| \sum_{k=0}^{j-1} T^k g \right\|_{L^p} \leq m(A)^{1/p} \left\| \sum_{k=0}^{j-1} T^k g \right\|\) which is uniformly bounded for \(j \geq 0\), applying Lemma 5 to the reflexive space \(L^p(m)\), we conclude that there exists \(f_0 \in L^p(m)\) such that \(f_0(\phi a) - f_0(a) = g(a)\). Since the mean ergodic theorem holds for \(T\) in the reflexive space \(L^p(m)\), \([8]\) the ergodic means \(n^{-1} \sum_{j=0}^{n-1} T^j f_0\) converges to an element \(f_1\) of \(L^p(m)\) in the strong topology of \(L^p(m)\) and \((I-T)f_1 = 0\). Let \(f = f_0 - f_1\). Then \(f(\phi a) - f(a) = g(a)\) a.e. while \(n^{-1} \sum_{j=0}^{n-1} T^j f \to 0\) in \(L^p(m)\) as \(n \to \infty\). Set \(h_n = n^{-1} \sum_{j=0}^{n-1} \sum_{k=0}^{j-1} T^k g\).
Then \( \|h_n\|_{L^\infty} \) are uniformly bounded for \( h_n = f - n^{-1} \sum_{j=1}^n T \) if converges in \( L^p(m) \) to \( f \) as \( n \to \infty \). Choosing a subsequence which converges to \( f \) a.e., it follows that \( f \in L^\infty(m) \).

3. Let \( A_1 \) and \( A_2 \) be two Hausdorff spaces, with \( A_1 \) a Baire space, i.e. of the second category on itself. Let \( \phi \) be a homeomorphism of \( A_1 \) onto itself such that \( A_1 \) is a minimal orbit closure under \( \phi \). Let \( \psi \) be a continuous mapping from \( A_1 \times A_2 \) into \( A_2 \). We shall consider functions \( h \) from \( A_1 \) to \( A_2 \) which satisfy the condition

\[
(3) \quad h(a_1) = \psi(a_1, h(a_1)), \quad a_1 \in A_1.
\]

The function \( h \) will be said to be a Baire function if there exists a set \( S \) of the first category in \( A_1 \) such that \( h \) is a continuous mapping of \( A_1 - S \) into \( A_2 \). If \( A_2 \) is a metric space, this definition includes all functions obtained by a sequence of pointwise sequential limits starting with continuous functions [2, Exercise 14, p. 81].

A family \( H \) of homeomorphisms of \( A_2 \) is said to be universally transitive if for every pair of distinct points \( a_2, a'_2 \) in \( A_2 \) there is a \( \xi \) in \( H \) such that \( \xi a_2 = a'_2 \).

**Theorem 3.** Let \( h \) be a Baire function from \( A_1 \) to \( A_2 \) for which (3) holds outside some set \( S_1 \) of first category in \( A_1 \). Suppose that \( A_2 \) is compact and that there exists a universally transitive family \( H \) of homeomorphisms of \( A_2 \), each of which has no fixed points and commutes with \( \psi \). Then after change on a set of the first category in \( A_1 \), \( h \) can be made into a continuous function from \( A_1 \) to \( A_2 \) satisfying (3) for all \( a_1 \) in \( A_1 \).

**Proof.** Let \( S_0 = \bigcup_{n \geq 0} \{ \phi^n(S) \cup \phi^n(S_1) \} \). Since \( \phi \) is a homeomorphism, \( S_0 \) is of first category in \( A_1 \). \( A_1 - S_0 \) is an invariant set with respect to \( \phi \) and dense in \( A_1 \), \( h \) is continuous from \( A_1 - S_0 \) to \( A_2 \), and (3) holds for all \( a_1 \) in \( A_1 - S_0 \). If we set \( B = A_2 \) in the discussion of §1 and \( \pi(a_1, a_2) = (\phi a_1, \psi(a_1, a_2)) \), the results of Lemmas 1, 2, and 3 are valid for \( \pi \). Let \( a'_1 \) be a point of \( A_1 - S_0 \), \( a'_2 = h(a'_1) \), \( F_0 \) a minimal invariant set contained in \( F(a'_1, a'_2) \). The condition (3) on \( h \) in \( A_1 - S_0 \) implies since \( h \) is continuous on \( A_1 - S_0 \), that if \( G \) is the graph of \( h \) on \( A_1 - S_0 \), then \( G = F(a'_1, a'_2) \cap \pi_{a_1}^{-1}(A_1 - S_0) \). We shall show that \( F_0 = F(a'_1, a'_2) \) and that for each \( a_1 \) in \( A_1 \), \( \pi_{a_1}^{-1}(a_1) \cap F_0 \) consists of a single point. The function \( f \) whose graph is \( F_0 \) will then be the desired continuous extension of \( h \).

It suffices to show that if \( (a_1, a_2) \) and \( (a_1, a_2^* \) lie in \( F_0 \), then \( a_2 = a_2^* \). If not, there is a homeomorphism \( \xi \in H \) commuting with \( \psi \) without fixed points on \( A_2 \) such that \( \xi a_2 = a_2^* \). By Lemma 3, however \( S_1 F_0 = F_0 \). Since \( \xi \) has no fixed points, \( S_1 \) has no fixed points. But then \( F_0 \) and a fortiori \( F(a'_1, a'_2) \) would have at least two points over
every point of \( A_1 \). Since over the points of \( A_1 - S_0 \), it has only one point, this is impossible.

We may specialize Theorem 3 in two ways: (1) by letting \( A_2 \) be a compact group, \( \psi_0 \) a mapping of \( A_1 \) into \( A_2 \), \( \psi(a_1, a_2) = \psi_0(a_1) \cdot a_2 \), the homeomorphism family \( \mathcal{H} \) be the elements of \( A_2 - \{e\} \) acting by right multiplication on \( A_2 \); (2) by letting \( A_2 \) be a compact space, \( \psi_0 \) be a homeomorphism of \( A_2 \) onto itself such that the group generated by \( \psi_0 \) is equicontinuous, \( \psi(a_1, a_2) = \psi_0(a_2) \), \( \mathcal{H} \) the closure of the group of homeomorphisms generated by \( \psi_0 \) except for the identity. In this second case we may replace \( A_2 \) by the orbit closure under \( \psi_0 \) of one of the values taken by \( h \) on an element of \( A_1 - S_0 \). It is known [5, 9.33, pp. 78–79] that on this orbit closure \( \mathcal{H} \) is universally transitive and, unless the orbit closure is finite, the elements of \( \mathcal{H} \) have no fixed points on this set. If we modify \( h \) to make it a continuous mapping into this set, it will be a continuous mapping into \( A_2 \).

In these two cases the specialized forms of Theorem 3 become:

**Theorem 4.** Let \( A_1 \) be a Baire space, \( A_2 \) a compact group, \( \phi \) a homeomorphism of \( A_1 \) onto itself such that \( A_1 \) is a minimal orbit closure under \( \phi \). Let \( \psi_0 \) be a continuous mapping of \( A_1 \) into \( A_2 \). Suppose that the Baire function \( h \) satisfies the relation

\[
h(\phi a_1) = \psi_0(a_2) \cdot h(a_1)
\]

for all \( a_1 \) outside a set of the first category in \( A_1 \). Then after change on a set of the first category in \( A_1 \), \( h \) can be made into a continuous function from \( A_1 \) to \( A_2 \) which satisfies (1) for all \( a_1 \in A_1 \).

**Theorem 5.** Let \( A_1 \) be a Baire space, \( A_2 \) a compact space, \( \phi \) a homeomorphism of \( A_1 \) onto itself under which \( A_1 \) is a minimal orbit closure, \( \psi_0 \) a homeomorphism of \( A_2 \) onto itself which generates an equicontinuous group of homeomorphisms of \( A_2 \). Suppose that the Baire function \( h \) from \( A_1 \) to \( A_2 \) satisfies the relation

\[
h(\phi a_1) = \psi_0(h a_1)
\]

for all \( a_1 \) outside a set of the first category in \( A_1 \). Then after change on a set of the first category in \( A_1 \), \( h \) can be made into a continuous function from \( a_1 \) to \( a_2 \) satisfying (1)' for all \( a_1 \) in \( A_1 \).

**Bibliography**

ON SPACES WHICH ARE NOT OF COUNTABLE CHARACTER

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It is well known that the unit interval $I$ has a countable base and the fixed point property. By considering the maps $g(x) = x^2$ and $h(x) = 1 - x$, one sees that there is no $x \in I$ such that for every continuous map $f: I \to I$, $x \in f(I)$ implies $f(x) = x$.

In Theorem 1, it is shown that if $A$ is a closed, non-null proper subset of a locally connected, compact Hausdorff space $X$ which has a countable base, then there exists a continuous map $f: X \to X$ such that $A \cap f(X)$ is not contained in $A \cap f(A)$. Theorem 2 shows that certain nondegenerate topological spaces $X$ contain proper subsets $M$ such that for every continuous map $f: X \to X$, $M \cap f(X) \subseteq M \cap f(M)$. That is, for each of these spaces $X$ and every continuous map $f: X \to X$, $x \in M \cap f(X)$ implies $f^{-1}(x) \cap M \neq \emptyset$. The corollary is of interest in that, if $X$ satisfies the hypotheses of Theorem 2 and $M$ consists of a single point, then a fixed point of some of the maps $f: X \to X$ is located.

**Theorem 1.** Suppose $X$ is a connected, locally connected, compact Hausdorff space which has a countable base. If $A$ is any non-null, closed, proper subset of $X$, then there exists a continuous map $f: X \to X$ such that $A \cap f(X) \setminus A \cap f(A) \neq \emptyset$.

**Proof.** Since $X$ is compact Hausdorff and has a countable base, $X$ is metrizable. Hence $X$ is arcwise connected. Let $y \in X \setminus A$. Since...