ON NILSTABLE ALGEBRAS

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1. Introduction. A simple commutative power-associative algebra \( \mathfrak{A} \) of degree 2 over a field \( \mathbb{F} \) of characteristic not 2 has a unity element 
\[ 1 = u + v \]
where \( u \) and \( v \) are orthogonal idempotents. Then \( \mathfrak{A} \) may be 
decomposed relative to \( u \) and written as 
\[ \mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_{12} + \mathfrak{A}_2 \]
with 
\[ \mathfrak{A}_1 = \mathfrak{A}_u(1) = \mathfrak{A}_v(0), \quad \mathfrak{A}_{12} = \mathfrak{A}_u(1/2) = \mathfrak{A}_v(1/2) \]
and 
\[ \mathfrak{A}_2 = \mathfrak{A}_u(0) = \mathfrak{A}_v(1) \]
where 
\( x \) is in \( \mathfrak{A}_u(\lambda) \) if and only if \( xu = \lambda x \). Furthermore 
\[ \mathfrak{A}_1 = u\mathfrak{F} + \mathfrak{G}_1 \]
and 
\[ \mathfrak{A}_2 = v\mathfrak{F} + \mathfrak{G}_2 \]
where \( \mathfrak{G}_1 \) and \( \mathfrak{G}_2 \) are nilalgebras. It is known that \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) are orthogonal subalgebras of \( \mathfrak{A} \), 
\[ \mathfrak{A}_{12} \subseteq \mathfrak{A}_1 + \mathfrak{A}_2, \]
and 
\[ \mathfrak{A}_{12} \subseteq \mathfrak{A}_{12} + \mathfrak{G}_{1-i} \]
for \( i = 1, 2 \). Albert has defined \( u \) to be a stable idempotent and \( \mathfrak{A} \) to be \( u \)-stable in case 
\[ \mathfrak{A}_{12} \subseteq \mathfrak{A}_{12} \]
for \( i = 1, 2 \). We generalize this notion and call \( u \) a nilstable idempotent and \( \mathfrak{A} \) nilstable with 
respect to \( u \) if \( \mathfrak{A}_{12} \subseteq \mathfrak{A}_{12} + \mathfrak{G}_{1-i} \) for \( i = 1, 2 \). Thus every stable idempotent is also nilstable. It is known that every commutative power- 
associative algebra of degree 2 and characteristic 0 is nilstable with 
respect to every idempotent.\(^2\)

The purpose of this note is to give the proof of the following theorem.

THEOREM 1. Let \( \mathfrak{A} \) be a simple commutative power-associative algebra 
of degree 2 over a field \( \mathbb{F} \) whose characteristic is prime to 6. Then \( \mathfrak{A} \) is a 
Jordan algebra if and only if \( \mathfrak{A} \) is nilstable with respect to two idem-
potents \( u, f \) such that \( u \neq 1, f \neq 1, u + f \neq 1 \) and such that \( f \) is not of the 
form \( f = u + w_{12} + w_1 + w_2 \) or \( f = v + w_{12} + w_1 + w_2 \) with \( w_{12} \) in \( \mathfrak{A}_{12} \), \( w_1 \) in 
\( \mathfrak{G}_1 \), \( w_2 \) in \( \mathfrak{G}_2 \).

Since any Jordan algebra is stable with respect to each of its 
idempotents, we are only concerned with the other half of the theo-
rem. The proof is to a large extent the proof in [8] of the result that 
every simple commutative power-associative algebra of degree 2 and 
characteristic 0 is a Jordan algebra. Since we shall lean rather heavily 
on [8], we shall refer the reader to that paper instead of repeating 
those results here.

The simple \( u \)-stable algebras have been determined by Albert 
[3; 4; 5]. There remains the problem of finding all algebras which

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\(^1\) The above results are given in [1, 2, and 7].

\(^2\) See [7].
are not \( u \)-stable. It is hoped that Theorem 1 will be useful in solving the intermediate problem of determining the nilstable algebras.

2. Idempotents. Let \( \mathfrak{A} \) be nilstable with respect to an idempotent \( u \). Then if \( \mathfrak{A} \) has characteristic not 2 or 3, all the results of §2 of [8] are valid here. This is because the assumption of nilstability is the assumption of the conclusion of Lemma 1 of [8]. Characteristic not 2 is needed from the outset and it is necessary to divide by 3 at the end of the proof of Lemma 3.

A result needed to prove Theorem 1 is

**Theorem 2.** Let \( \mathfrak{A} \) be a commutative power-associative algebra of degree 2 and let \( u \) be a nilstable idempotent of \( \mathfrak{A} \). If \( f \) is any idempotent of \( \mathfrak{A} \) other than \( u \), \( v \), 1, then \( f = 1/2(1 + w) \) where \( w^2 = 1 \) and \( w = \gamma(u - v) + w_1 + w_2 \) where \( w_1 \neq 0 \) is in \( \mathfrak{A}_{12} \), \( w_i \) is in \( \mathfrak{O}_i \), \( i = 1, 2 \).

If \( f \) is any idempotent, then \( w = 2f - 1 \) has the property \( w^2 = 1 \). Then \( w = \gamma u + \delta v + w_1 + w_2 \) where \( \gamma, \delta \) are in \( \mathfrak{A} \), \( w_1 \) in \( \mathfrak{A}_{12} \), \( w_1 \) in \( \mathfrak{O}_1 \), \( w_2 \) in \( \mathfrak{O}_2 \). When \( w_1 = 0 \), \( w^2 = 1 = \gamma^2 u + \delta^2 v + 2\gamma w_1 + 2\delta w_2 + w_1^2 + w_2^2 \). Consequently, \( \gamma^2 = \delta^2 = 1 \), \( 2\gamma w_1 + w_2 = 0 \), \( 2\delta w_2 + w_2^2 = 0 \). Since \( \gamma \neq 0 \), \( w_1^2 = -2\gamma w_1 \) implies \( (1/2\gamma)w_1^2 = -(1/2\gamma)w_1 \) so \(- (1/2\gamma)w_1 \) is idempotent or zero. But \( w_1 \) is nilpotent so \( w_1 = 0 \). Similarly, \( w_2 = 0 \). Thus \( w = \gamma u + \delta v \) where \( \gamma = \pm 1 \), \( \delta = \pm 1 \). It follows that \( w = 1, -1, u - v \), or \( v - u \), and \( f = 1/2(1 + w) = 0, 1, u, \) or \( v \). By hypothesis \( f \) is not 0, 1, \( u \), or \( v \), so \( w_1 \) must be nonzero.

Computing \( w^2 = 1 \) we have \( \gamma^2 u + \delta^2 v + (\gamma + \delta)w_1 + w_2^2 + 2w_1(1 + w) + 2\gamma w_1 + 2\delta w_2 + w_1^2 + w_2^2 = 1 \). Equating components in \( \mathfrak{A}_{12} \), \( (\gamma + \delta)w_1 + 2w_1[S_{1/2}(w_1) + T_{1/2}(w_2)] = 0 \) where \( w_1[2[S_{1/2}(w_1) + T_{1/2}(w_2)] \) is the component of \( w_1(w_1 + w) \) in \( \mathfrak{A}_{12} \). It is known [2, page 517; and 7] that \( S_{1/2}(w_1) + T_{1/2}(w_2) \) is a nilpotent mapping. Since \( w_1 \neq 0 \), it follows that \( \gamma + \delta = 0 \), as desired.

3. Proof of Theorem 1. By hypothesis \( \mathfrak{A} \) is nilstable with respect to \( u \) and \( f \), and, by Theorem 2, \( f = 1/2(1 + w) \), \( w = \gamma(u - v) + w_1 + w_2 \). If \( \gamma = 0 \), the proof of the theorem of reference [8] gives us the result of Theorem 1. In making the induction of §4 of [8], we consider only the class of algebras satisfying the hypotheses of Theorem 1. Thus we may now assume \( \gamma \neq 0 \). Even in this case the proof is patterned after that of reference [8]. Lemma 8 of [8] holds in our situation.

Next we proceed to derive a result comparable to Lemma 9 of [8]. Any element of \( \mathfrak{A}(\lambda) \) has the form \( a = \alpha u + \beta v + a_1 + a_2 \) where \( \alpha, \beta \) are in \( \mathfrak{A} \), \( a_1 \), \( a_2 \) are in \( \mathfrak{O} = \mathfrak{O}_1 + \mathfrak{O}_2 \), \( a_1 \) in \( \mathfrak{A}_{12} \). Then \( wa = \alpha\gamma u - \beta\gamma v + \gamma(a_1 - a_2) + 1/2(\alpha + \beta)w_1 + w_{12}(a_1 + a_2) + \alpha w_1 + \beta w_2 \).
+ a_{12}(w_1 + w_2) + a_1w_1 + a_2w_2 = (2\lambda - 1)(\alpha u + \beta v + a_{12} + a_1 + a_2).

If \lambda = 1, Lemma 10 of [2] implies \(w_{12}a_{12} = \alpha(1 - \gamma) = \beta(1 + \gamma)\) where by \(w_{12}a_{12} = \alpha(1 - \gamma)1\) we mean \(w_{12}a_{12} = \alpha(1 - \gamma)1\) is in \(\mathcal{O}\). From \(\alpha(i - \gamma) = \beta(1 + \gamma)\) we get \(2\alpha = \alpha + \alpha\gamma + \beta(1 + \gamma) = (\alpha + \beta)(1 + \gamma)\) or \(\alpha(1 - \gamma) = (\alpha + \beta)(1 - \gamma^2)/2 \equiv w_{12}a_{12}\). Also, \(\omega_{12}[S_{1/2}(a_1) + T_{1/2}(a_2)] + a_{12}[S_{1/2}(w_1) + T_{1/2}(w_2)] = a_{12}\). Multiply both sides by \(w_{12}T_{1/2}(g_2)\), use the fact \(w_{12}[S_{1/2}(w_1) + T_{1/2}(w_2)] = 0\) which follows from our computation of \(w^2 = 1\), and use the results of \([8, \S 2]\). Thus \(b_{12} \cdot w_{12}T_{1/2}(g_2)\) is in \(\mathcal{O}\) for any \(g_2\) in \(\mathcal{O}\). If \(\lambda = 0\), the calculations yield \(b_{12}a_{12} = -2^{-1}(\alpha + \beta) \cdot (1 - \gamma^2) = -\alpha(1 + \gamma) = -\beta(1 - \gamma)\). When \(\lambda = 1/2\), \(b_{12}a_{12} = -\alpha\gamma = \beta\gamma\) so \(\gamma(\alpha + \beta) = 0\) and since \(\gamma \neq 0\), \(\beta = -\alpha\). Some of these results are formally stated in the following lemma.

**Lemma 1.** Any element \(a\) of \(\mathcal{U}\) may be written \(a = \alpha u + \beta v + a_{12} + a_1 + a_2\) with \(\alpha, \beta\) in \(\mathcal{F}\), \(a_{12}\) in \(\mathcal{U}_{12}\), \(a_1\) in \(\mathcal{O}_1\), \(a_2\) in \(\mathcal{O}_2\), respectively. If \(a\) is in \(\mathcal{U}_{f}(\lambda)\), \(\lambda = 0, 1\), then \(b_{12} \cdot w_{12}T_{1/2}(g_2)\) is in \(\mathcal{O}\) for any \(g_2\) in \(\mathcal{O}\). When \(a\) is in \(\mathcal{U}_{f}(1/2)\), \(\beta = -\alpha\).

Now let \(a\) be in \(\mathcal{U}_{f}(1)\) and let \(b = \xi u + \eta v + b_{12} + b_1 + b_2\) also be in \(\mathcal{U}_{f}(1)\). Then \(ab = \alpha\xi u + \beta\eta v + 2^{-1}(\alpha + \beta)b_{12} + 2^{-1}(\xi + \eta)a_{12} + a_{12}b_{12} + a_{12}(b_1 + b_2) + b_{12}(a_1 + a_2) + \alpha b_1 + \beta b_2 + \xi a_1 + \eta a_2 + a_1b_1 + a_2b_2\) and \(ab\) is in \(\mathcal{U}_{f}(1)\). Let \(ab = \theta u + \phi v + c_1 + c_2\) and let \(a_{12}b_{12} = \rho.\) We have \(\theta = \alpha\xi + \rho, \phi = \beta\eta + \rho.\) From the results above Lemma 1, we have \((\alpha\xi + \rho)(1 - \gamma) = (\beta\eta + \rho)(1 + \gamma)\). Also, \(\xi(1 - \gamma) = \eta(1 + \gamma), \beta(1 + \gamma) = \alpha(1 - \gamma)\). Therefore, \(\alpha\eta(1 + \gamma) + \rho(1 - \gamma) = \alpha\eta(1 - \gamma) + \rho(1 + \gamma)\), and \(2\alpha\eta\gamma = 2\rho\gamma\). Since \(\gamma \neq 0\), \(\rho = \alpha\eta\) and \(\theta = \alpha(\xi + \eta), \phi = \eta(\alpha + \beta)\).

**Lemma 2.** Let \(a = \alpha u + \beta v + a_{12} + a_1 + a_2\) and \(b = \xi u + \eta v + b_{12} + b_1 + b_2\) be any two elements of \(\mathcal{U}_{f}(1)\). Then if \(\gamma \neq 0\), \(a_{12}b_{12} = \alpha\eta\) and \(ab = \alpha(\xi + \eta)u + \eta(\alpha + \beta)v + c_1 + c_2\).

**Corollary 1.** Let \(a = \alpha u + \beta v + a_{12} + a_1 + a_2\) be any element of \(\mathcal{U}_{f}(1)\). Then \(a^k = \alpha(\alpha + \beta)^{k-1}u + (\alpha + \beta)^{k-1}v + c_1 + c_2\).

**Lemma 3.** If \(a\) is nilpotent, then \(\alpha = \beta = 0\).

The corollary to Lemma 2 implies that if \(a\) is nilpotent \(\alpha = 0, \beta = 0\) or \(\alpha + \beta = 0\). If \(\beta = -\alpha\), the fact that \(\alpha(1 - \gamma) = \beta(1 + \gamma)\) implies \(\alpha = \beta = 0\).

The results of Lemma 2, its corollary, and Lemma 3 can be proved in the same manner for elements of \(\mathcal{U}_{f}(0)\).

**Lemma 4.** Let \(a = \alpha u + \beta v + a_{12} + a_1 + a_2\) be any element in \(\mathcal{U}_{f}(1)\) or \(\mathcal{U}_{f}(0)\) and \(c = \delta(u - v) + c_1 + c_2\) be any element of \(\mathcal{U}_{f}(1/2)\). Then \(a_{12}c_{12} = 2^{-1}(\beta - \alpha)\delta.\) If \(a\) is nilpotent, \(a_{12}c_{12} = 0.\)
The product \( ac = m + n \) where \( m \) is in \( \mathcal{A}_f(1/2) \) and \( n \) is in \( \mathcal{A}_f(1 - \lambda) \) when \( a \) is in \( \mathcal{A}_f(\lambda) \), \( (\lambda = 0, 1) \). By assumption \( \mathcal{A} \) is nilstable with respect to \( f \) so \( n \) is nilpotent. Lemma 3 and the corresponding result for elements in \( \mathcal{A}_f(0) \) imply \( n = n_{12} + n_1 + n_2 \) with \( n_1, n_2 \) in \( \mathcal{G} \). By Lemma 1, \( m = \mu(u-v) + m_{12} + m_1 + m_2 \). From the equation \( ac = m + n \) we obtain \( \alpha \delta u - \beta \delta v + a_{12}c_{12} \equiv \mu(u-v) \) and it follows that \( \mu - \alpha \delta = -\mu + \beta \delta, \mu = 2^{-1}(\alpha + \beta) \delta, \alpha_{12}c_{12} = 2^{-1}(\beta - \alpha) \delta \).

As in [8], consider any element \( g_2 \) in \( \mathcal{G}_2 \) and write \( g_2 = g_f(1) + g_f(0) + g_f(1/2) \) where \( g_f(\lambda) \) is in \( \mathcal{A}_f(\lambda) \). Write each \( g_f(\lambda) \) as a sum of elements determined by the decomposition of \( \mathcal{A} \) relative to \( u \). Let \( g_f(1) = \alpha u + \beta v + a_{12} + a_1 + a_2 \), \( g_f(0) = \xi u + \eta v + b_{12} + b_1 + b_2 \), \( g_f(1/2) = \phi(u-v) + d_{12} + d_1 + d_2 \). Then \( \alpha + \xi = -\phi = -\beta - \eta \). The results before Lemma 1 imply \( \alpha(1 - \gamma) = \beta(1 + \gamma) \) and \( \xi(1 + \gamma) = \eta(1 - \gamma) \). Subtract the second from the first of these relations to get \( \alpha - \xi = \beta - \eta \). Add corresponding sides of this relation to \( \alpha + \xi = -\beta - \eta \) so that \( \alpha = -\eta \), and then \( \beta = -\xi \). Now by Lemma 4 and for any \( c_{12} \) belonging to \( \mathcal{A}_f(1/2) \), \( (a_{12} - b_{12})c_{12} = 2^{-1}(\beta - \alpha) \delta - 2^{-1}(\eta - \xi) \delta = 0 \). By Lemma 8 of [8], \( a_{12} - b_{12} = w_{12}T_{1/2}(g_2) \). Thus we have proved Lemma 12 and Theorem 1 of [8].

The induction of §4 of [8] made in the class of algebras nilstable with respect to two idempotents \( u, f \) such that \( u \neq 1, f \neq 1, u + f \neq 1 \), \( f \neq u + z_{12} + z_1 + z_2 \), and \( f \neq v + z_{12} + z_1 + z_2 \) with \( z_{12} \) in \( \mathcal{A}_{12} \), \( z_1 \) in \( \mathcal{A}_1 \), \( z_2 \) in \( \mathcal{A}_2 \), and the induction completes the proof of Theorem 1. In order to successfully complete the induction it is necessary to have \( w_{12}^2 \) non-nilpotent. Since \( w_{12}^2 \equiv (1 - \gamma^2) \), this means we need to have \( \gamma^2 - 1 \neq 0 \), \( \gamma \neq \pm 1 \). The condition \( \gamma = \pm 1 \) implies

\[
f = (1 + w)/2 = u + 2^{-1}(w_{12} + w_1 + w_2)
\]
or \( f = v + 2^{-1}(w_{12} + w_1 + w_2) \).

4. The cases \( \gamma = \pm 1 \). The result of Theorem 1 is not true when \( \gamma = \pm 1 \). For example consider the \( u \)-stable algebra \( \mathcal{G} \) of characteristic \( p > 5 \) described in [6]. It is not a Jordan algebra and \( \mathcal{G} = u\mathcal{A} + v\mathcal{A} + y_0\mathcal{A} + y_1\mathcal{A} \) where \( \mathcal{A} = \mathfrak{F} [1, x] \), \( x^p = 0 \). In the decomposition relative to \( u \), \( \mathcal{G}_{12} = y_0\mathcal{A} + y_1\mathcal{A}, \mathcal{G}_1 = u\mathcal{A}, \mathcal{G}_2 = v\mathcal{A} \). Let \( w = u - v + 2y_1x^{-1} \) so that \( f = 2^{-1}(1+w) = u + y_1x^{-1} \). If \( a = a(u + \beta v + a_{12} + a_1 + a_2) \). The proof of Lemma 1 implies \( \alpha(1 - \gamma) = \beta(1 + \gamma) \). Since \( \gamma = 1, \beta = 0 \). Furthermore \( wa = a, \alpha u + a_1 - a_2 + \alpha y_1x^{-1} + 2y_1x^{-1} \cdot a_{12} = \alpha u + a_1 + a_2 + a_2 \) where we have used the fact that \( y_1x^{-1}(a_1 + a_2) = 0 \). It follows that \( a_{12} = \alpha y_1x^{-1} \) and hence \( a_2 = 0 \). Thus \( a = \alpha u + \alpha y_1x^{-1} + a_1 = \alpha f + a_1 \) where \( a_1 \) is any nilpotent element of \( \mathcal{G}_1 \). Similarly, if \( b \) is any element of \( \mathcal{G}_f(0) \), \( b = \beta v - \beta y_1x^{-1} + b_2 = \beta(1-f) + b_2 \) where \( b_2 \) is any nilpotent
element of \( \mathcal{S}_2 \). Next let \( c \) be any element of \( \mathcal{S}_f(1/2) \). By Lemma 1, \( c = \lambda(u - v) + c_{12} + c_1 + c_2 \) and \( wc = 0 \). The product \( wc = 0 \) implies \( \lambda(u + v) + c_1 - c_2 + 2y_1x^{p-1} \cdot c_{12} = 0 \). The multiplication table of \( \mathcal{S} \) implies \( 2y_1x^{p-1} \cdot c_{12} \) is nilpotent and therefore it is equal to \( c_2 - c_1 \) and \( \lambda = 0 \). The product \( ca = (c_{12} + c_1 + c_2)(\alpha f + a_1) = 2^{-1}\alpha(c_{12} + c_1 + c_2) + (c_{12} + c_1) a_1 \). We know \( ca \) is in \( \mathcal{S}_f(1/2) + \mathcal{S}_f(0) \), so \( w(ca) \) is the negative of the component in \( \mathcal{S}_f(0) \). Computing, we get \( w(ca) = 2y_1x^{p-1} \cdot c_{12} a_1 + c_1 a_1 \).

By the definition of multiplication in \( \mathcal{S} \), \( y_1x^{p-1} \cdot c_{12} a_1 \) is 0 or a scalar multiple of \( x^{p-1} \). Since \( 2y_1x^{p-1} \cdot c_{12} = c_2 - c_1 \), \( c_1a_1 = -(2y_1x^{p-1} \cdot c_{12})a_1 \), which is a scalar multiple of \( x^{p-1}u \). Therefore, the component of \( ca \) in \( \mathcal{S}_f(0) \) is nilpotent. Similarly, \( cb \) is the sum of an element in \( \mathcal{S}_f(1/2) \) and a nilpotent element of \( \mathcal{S}_f(1) \). This proves that \( \mathcal{S} \) is nilstable with respect to \( f \), and shows that the restriction \( \gamma \neq \pm 1 \) is necessary in order to obtain the result of Theorem 1. In our example we took \( \gamma = 1 \), but we could just as easily let \( \gamma = -1 \), \( w = v - u + 2y_1x^{p-1} \), and \( f = v + y_1x^{p-1} \).

**References**


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