ON A CLOSURE PROPERTY OF MEASURABLE SETS

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1. Introduction. Let $X$ be a space. A real-valued set function $\Lambda(E)$ defined for all subsets $E$ of $X$ such that $0 \leq \Lambda(E) \leq \infty$ is called an outer measure if it satisfies the following conditions.

(i) $\Lambda(E) = 0$ if $E$ is the empty set.
(ii) $\Lambda(E_1) \leq \Lambda(E_2)$ if $E_1 \subseteq E_2$.
(iii) $\Lambda(E) \leq \Lambda(E_1) + \Lambda(E_2) + \cdots$ if $E = E_1 \cup E_2 \cup \cdots$.

A set $E$ is called $\Lambda$ measurable if for every pair of subsets $P, Q$ of $X$ with $P \subseteq E, Q \subseteq X - E$, the equality $\Lambda(P \cup Q) = \Lambda(P) + \Lambda(Q)$ holds. Let $\mathcal{M}(\Lambda)$ denote the family of $\Lambda$ measurable sets. $\mathcal{M}(\Lambda)$ is a completely additive class of sets and $\Lambda$ is a measure on $\mathcal{M}(\Lambda)$ (see, for example, Saks [2, pp. 45-45]). Numbers in square brackets refer to the bibliography at the end of this note. $\Lambda$ is called regular if every set is contained in a $\Lambda$ measurable set of equal $\Lambda$ measure. $\Lambda$ is called finite-valued if $\Lambda(X) < \infty$.

For each family $\mathcal{F}$ of subsets of $X$ let there be associated a family $\mathcal{E}(\mathcal{F})$ of subsets of $X$ satisfying the following conditions, (a) $\mathcal{F} \subseteq \mathcal{E}(\mathcal{F})$.
(b) If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $\mathcal{E}(\mathcal{F}_1) \subseteq \mathcal{E}(\mathcal{F}_2)$. It is the purpose of this note to establish the following closure property of measurable sets.

**Theorem.** Under the above conditions if the relationship $\mathcal{E}[\mathcal{M}(\Lambda)] = \mathcal{M}(\Lambda)$ holds for every finite-valued regular outer measure $\Lambda$ then it holds for every outer measure $\Lambda$.

As an application of this theorem let $\mathcal{E}(\mathcal{F})$ denote the family of set obtained from $\mathcal{F}$ by the operation $(A)$ (see Saks [2, p. 47], for the definition of this operation). It is well known that in this case $\mathcal{E}[\mathcal{M}(\Lambda)] = \mathcal{M}(\Lambda)$ for every outer measure $\Lambda$ but as noted in Saks [2] the proof is much simpler if $\Lambda$ is assumed to be regular (see, Kuratowski [1, p. 58]). Since $\mathcal{E}(\mathcal{F})$ in this case satisfies the above conditions (a) and (b), by the above theorem to show that $\mathcal{E}[\mathcal{M}(\Lambda)] = \mathcal{M}(\Lambda)$ holds for every outer measure $\Lambda$ it is sufficient to show that it holds for every finite-valued regular outer measure $\Lambda$.

2. Proof of the theorem. We first prove the following result.

**Lemma.** If $\Lambda$ is an outer measure in $X$ and $E^* \in \mathcal{M}(\Lambda)$ then there is a

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finite-valued regular outer measure $\Lambda^*$ in $X$ such that $\mathcal{M}(\Lambda) \subseteq \mathcal{M}(\Lambda^*)$ and $E^* \in \mathcal{M}(\Lambda^*)$.

**Proof.** Since $E^* \in \mathcal{M}(\Lambda)$ there is a pair of subsets $P^*, Q^*$ of $X$ such that $P^* \subseteq E^*$, $Q^* \subseteq X - E^*$ and

$$\Lambda(P^* \cup Q^*) < \Lambda(P^*) + \Lambda(Q^*) < \infty.$$  

Set $R^* = P^* \cup Q^*$ and for each subset $E$ of $X$ set

$$\Lambda^*(E) = \operatorname{gl.b.} \Lambda(H \cap R^*) \text{ for } E \subseteq H, H \in \mathcal{M}(\Lambda).$$

It follows easily that $\Lambda^*$ is an outer measure in $X$ and for $E \subseteq X$ there is an $H \in \mathcal{M}(\Lambda)$ such that $E \subseteq H$, $\Lambda^*(E) = \Lambda(H \cap R^*) = \Lambda^*(H)$.

For $E \in \mathcal{M}(\Lambda)$ let $P, Q$ be a pair of subsets of $X$ with $P \subseteq E$, $Q \subseteq X - E$. Let $H \in \mathcal{M}(\Lambda)$ be such that $P \cup Q \subseteq H$ and $\Lambda^*(P \cup Q) = \Lambda(H \cap R^*)$. Since $E \in \mathcal{M}(\Lambda)$, $P \subseteq H \cap E \in \mathcal{M}(\Lambda)$, $Q \subseteq H \cap (X - E) \in \mathcal{M}(\Lambda)$,

$$\Lambda^*(P \cup Q) = \Lambda(H \cap R^*) = \Lambda(H \cap E \cap R^*)$$

$$+ \Lambda[H \cap (X - E) \cap R^*] \geq \Lambda^*(P) + \Lambda^*(Q).$$

By (iii) of §1 the equality sign holds in (2). Thus $E \in \mathcal{M}(\Lambda^*)$ and $\mathcal{M}(\Lambda) \subseteq \mathcal{M}(\Lambda^*)$. From this fact it follows that $\Lambda^*$ is regular. From

$$\Lambda^*(X) = \Lambda(R^*) < \infty.$$  

Since $\Lambda^*(P^* \cup Q^*) = \Lambda(P^* \cup Q^*) < \Lambda(P^*) + \Lambda(Q^*) \leq \Lambda^*(P^*) + \Lambda^*(Q^*)$, it follows that $E^* \in \mathcal{M}(\Lambda^*)$.

The proof of the theorem stated in §1 is now immediate. Assume that $\mathcal{G}[\mathcal{M}(\Lambda)] = \mathcal{M}(\Lambda)$ for every finite-valued regular outer measure $\Lambda$. Let $\Lambda$ be an outer measure and let $E^*$ be a set not in $\mathcal{M}(\Lambda)$. By the preceding lemma there is a finite-valued regular outer measure $\Lambda^*$ such that

$$\mathcal{M}(\Lambda) \subseteq \mathcal{M}(\Lambda^*), \quad E^* \in \mathcal{M}(\Lambda^*).$$

Since $\mathcal{G}[\mathcal{M}(\Lambda^*)] = \mathcal{M}(\Lambda^*)$ by assumption, from (3) and condition (b) of §1, $\mathcal{G}[\mathcal{M}(\Lambda)] \subseteq \mathcal{G}[\mathcal{M}(\Lambda^*)] = \mathcal{M}(\Lambda^*)$ and $E^* \in \mathcal{G}[\mathcal{M}(\Lambda)]$. Since $E^*$ was any set not in $\mathcal{M}(\Lambda)$, $\mathcal{G}[\mathcal{M}(\Lambda)] \subseteq \mathcal{M}(\Lambda)$. Therefore, by condition (a) of §1, $\mathcal{G}[\mathcal{M}(\Lambda)] = \mathcal{M}(\Lambda)$.

**Bibliography**


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