AUTOMORPHISMS OF PRODUCTS OF MEASURE SPACES

DOROTHY MAHARAM

1. Introduction. Let $S$ be the measure-theoretic product of a (not necessarily countable) family of unit intervals, $S = \prod I_\alpha$, $\alpha \in A$. In this paper we shall prove that $S$ has the following "realization" property: every "set automorphism" $\phi$ of $S$ may be induced by some "point automorphism" $T$ of $S$. (For the terms used, see below.) The particular case of this theorem in which $\phi$ is measure-preserving shows that $S$ has "sufficiently many measure-preserving transformations" in the terminology of Halmos and von Neumann [1, p. 340].

When the number of factors $I_\alpha$ is countable, this theorem reduces, in the measure-preserving case at least, to a known property of normal measure spaces [2, p. 582]. The method of proof of the general theorem will apply, more generally, to any product of measure spaces in which (a) each factor has a separating sequence, and is of total measure 1, (b) every sub-product of countably many factors has the "realization" property itself. Thus, for example, any product of 2-point factors (of measure 1) will also have the realization property. We could even allow a finite number of the factors to have infinite (but $\sigma$-finite) measure, for the transformations considered need not preserve measure. However, we shall restrict attention to the theorem as first stated.

One feature needs remark. When the number of factors is countable, $T$ is uniquely determined by $\phi$, in the sense that if $T_1$ and $T_2$ are point-automorphisms which induce the same set-automorphism $\phi$, then $T_1$ and $T_2$ can differ on a null set at most. But when $S = \prod I_\alpha$, $\alpha \in A$, where $A$ is uncountable, $T$ is by no means unique in this sense. For example, if $T_1$ is the identity transformation on $S$ and $T_2$ the transformation which interchanges the coordinate values 0 and 1 wherever they occur, then $T_1$ and $T_2$ both induce the identity set-automorphism. But the set on which $T_1$ and $T_2$ differ can be shown to be nonmeasurable, having outer measure 1 and inner measure 0.

2. Notation. Let $S$ be any measure space, and $E$ its algebra of measurable sets modulo null sets. Thus if $X$ is a measurable subset of $S$, its class modulo null sets, denoted by $[X]$ or $x$, is a typical member of $E$. If $S'$ is another measure space, with $E'$ as its measure algebra,
a point isomorphism\(^1\) \(T\) from \(S\) to \(S'\) is a 1-1 mapping of \(S\) onto \(S'\) such that both \(T\) and \(T^{-1}\) take (i) measurable sets into measurable sets, (ii) null sets into null sets. (In the cases we are mainly concerned with, (ii) is a consequence of (i).) A set isomorphism\(^1\) \(\phi\) from \(S\) to \(S'\) is simply an isomorphism from \(E\) to \(E'\), that is, a 1-1 mapping of \(E\) onto \(E'\) which preserves suprema and complements, but not necessarily measure. Thus every \(T\) induces a \(\phi\) by the rule \(\phi(x) = \{ T(X) \}\), \(X \in x\). When \(S = S'\) and \(E = E'\) we speak of point and set automorphisms.

Throughout what follows, we assume \(S = \prod I_\alpha, \alpha \in A\), where each \(I_\alpha\) is a unit interval of real numbers. For each nonempty \(B \subset A\), we write \(S(B)\) for the partial product \(\prod I_\alpha, \alpha \in B\), using \(S(B)\) to denote both the product set and the measure space on it. For simplicity of notation, we also disregard the order of the factors, writing e.g. \(S = S(A) = S(B) \times S(A - B)\). \(\emptyset\) is used for the empty set, and we assume throughout that \(A \neq \emptyset\).

If \(C \subset B \subset A\), the “projection” \(\pi_{BC}: S(B) \to S(C)\) is defined as usual by \(\pi_{BC}(p) = q\), where \(p \in S(B)\) and the \(\alpha\)th coordinate \(q_\alpha\) of \(q\) is \(p_\alpha\), \(\alpha \in C\). When \(B = A\), \(\pi_{AC}\) is abbreviated to \(\pi_C\).

\(S^B\) denotes the family of “cylinders” on the measurable subsets of \(S(B)\), i.e., of sets \(\pi_{B}^{-1}(X) = X \times S(A - B)\) where \(X\) is a measurable subset of \(S(B)\). The algebra of measurable sets modulo null sets of \(S(B)\) will be written \(E(B)\), and that of \(S^B\) (modulo null sets of \(S(A)\)) will be written \(E^B\). It is well known (but not completely trivial) that \(\pi_B\) induces a measure-preserving isomorphism, which we denote by \(\pi_B\) also, from \(E^B\) to \(E(B)\).

3. Some lemmas.

**Lemma 1.** Let \(E_1, E_2\) be the measure algebras of two \(\sigma\)-finite measure spaces \(S_1, S_2\), and let \(E_3\) be the measure algebra of \(S_1 \times S_2\). Then, given automorphisms \(\psi_1, \psi_2\) of \(E_1, E_2\), there is a unique automorphism \(\psi_3\) of \(E_3\) such that

\[
\psi_3(x \times y) = \psi_1(x) \times \psi_2(y) \quad (x \in E_1, \ y \in E_2).
\]

The units \(e_1, e_2\), of \(E_1, E_2\), may be partitioned into disjoint elements \(a_{1n} \in E_1, \ a_{2n} \in E_2 \ (n = 1, 2, \cdots)\), of finite measure, such that whenever \(y \leq a_{in} \ (i = 1, 2)\) we have \((1/n)\) meas \(y \leq \ meas \psi(y) \leq n \ meas \ y\). It is a routine matter to extend the correspondence

\[
\psi_3(x \times y) = \psi_1(x) \times \psi_2(y) \quad (x \leq a_{1m}, \ y \leq a_{2n})
\]

\(^1\) This terminology differs from that in [1], where isomorphisms are required to be measure-preserving.
to an isomorphism between the ideals $A_{mn}$ and $B_{mn}$ of $E_3$, where $A_{mn}$ consists of all elements $\leq a_{1m} \times a_{2n}$ and $B_{mn}$ of all elements $\leq \psi_1(a_{1m}) \times \psi_2(a_{2n})$, and thence to extend $\psi_3$ to all of $E_3$. The proof that $\psi_3$ has the stated property, and of its uniqueness, presents no difficulty.

**Lemma 2.** Let $\phi$ be a set automorphism of $S = \prod I_a$, and let $T$ be a 1-1 mapping of $S$ onto itself such that, for each finite set $C$ of suffixes $\alpha$, and for each measurable set $K$ of $S^C$, $T(K) \in \phi\{K\}$ and $T^{-1}(K) \in \phi^{-1}\{K\}$. Then $T$ is a point automorphism of $S$, and induces $\phi$.

Let $\mathcal{B}$ be the Borel field generated by all sets of the form $K$, i.e., by all cylinder sets which are based on measurable sets in finite products of $I_a$'s. We recall that $\mathcal{B}$ generates $S$ in the following sense: (1) each measurable subset of $S$ differs from some set in $\mathcal{B}$ by a null set, (2) each null subset of $S$ is contained in some null set in $\mathcal{B}$. Now it is easy to see that the measurable subsets $X$ of $S$ which have the property:

$$ T(X) \in \phi\{X\} \quad \text{and} \quad T^{-1}(X) \in \phi^{-1}\{X\}, $$

form a Borel field. Hence every set in $\mathcal{B}$ has this property. From (2) it follows that $T(X)$ and $T^{-1}(X)$ are null whenever $X$ is null, and hence (1) shows that every measurable $X$ has the property. In particular, $T(X)$ and $T^{-1}(X)$ are measurable if $X$ is, so $T$ is a point automorphism of $S$; and clearly $T$ induces $\phi$.

**Definition.** Let $\phi$ be a given automorphism of $E$. A set $B \subset A$ will be called "invariant" (under $\phi$) if $\phi(B) = B$. Restricted to $E^B$, $\phi$ will then be an automorphism of $E^B$.

**Lemma 3.** Each countable set $B \subset A$ is contained in some countable subset $\overline{B}$ of $A$ which is invariant.\(^2\)

Let $B_0 = B$, and take a countable basis $b_{0m}$, $m = 1, 2, \cdots$, for $E^{B_0}$ (apply $\pi_{B_1}^{-1}$ to a countable basis for $E(B)$). Consider the elements $\phi^n(b_{0m})$ ($n = 0, \pm 1, \pm 2, \cdots$). The properties (1) and (2) stated at the beginning of the proof of Lemma 2, show that each of these measure classes contains a set which is a cylinder on only countably many coordinates; hence there is a countable set $B_1 \subset A$ such that every $\phi^n(b_{0m})$ is in $E^{B_1}$. Take a countable basis $b_{1m}$, $m = 1, 2, \cdots$, for $E^{B_1}$, and repeat the process, obtaining a countable set $B_2$; and so on. Then $\bigcup B_k$ ($k = 0, 1, \cdots$) is the countable invariant set required.

**Lemma 4.** Every set-automorphism of the unit interval can be induced by a point-automorphism.\(^2\)

\(^2\) More generally, each $B \subset A$ is contained in an invariant set $\overline{B} \subset A$ of cardinal $\leq \max (\mathfrak{N}_0, |B|)$. It can be shown that, given $B (\neq \emptyset)$, there is a smallest $\overline{B}$. 

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Let $\phi$ be a set-automorphism of the unit interval $I$, and for each $t \in I$ let $I_t$ denote the interval from 0 to $t$. The mapping $U$ defined by $U(t) = \text{meas } \phi \{ I_t \}$ is a point-automorphism of $I$, and induces a set-automorphism $\psi$. Since $U$ maps $I_t$ onto the interval from 0 to $U(t)$, we have $\text{meas } \psi \{ I_t \} = \text{meas } \phi \{ I_t \}$ for each $t \in I$; it follows that $\psi(x)$ and $\phi(x)$ have the same measure for each class $x$ in the measure algebra of $I$. Thus $\psi^{-1}\phi$ is a measure-preserving set-automorphism of $I$, and [2] there is a point-automorphism $V$ of $I$ which induces $\psi^{-1}\phi$. Then $UV$ is a point-automorphism of $I$ which induces $\phi$.

**Lemma 5.** Let $S = \prod I_\alpha$, $\alpha \in A$, and let $B$ be any subset of $A$ for which $A - B$ is countable. Suppose $\phi$ is a set-automorphism of $S$ which, restricted to $E^B$, is the identity mapping of $E^B$. Then there exists a point-automorphism $T$ of $S$ which induces $\phi$, and which satisfies $\pi_B T = \pi_B$.

Note that the hypothesis on $\phi$ implies that $B$ is invariant.

Write $C = A - B$; by Lemma 3, $C \subseteq \overline{C} \subseteq A$ where $\overline{C}$ is countable and invariant. Writing $D = \overline{C} - C$, we have $D \subseteq B$. Now, since $\phi(E^\overline{C}) = E^\overline{C}$, $\phi$ induces an automorphism $\phi_1 = \pi_{\overline{C}} \phi \pi_{\overline{C}}^{-1}$ on $E(\overline{C})$. And, because $\overline{C}$ is countable, $S(\overline{C})$ is isomorphic, under a measure-preserving point-isomorphism, to $I$. Hence, by Lemma 4, there exists a point-automorphism $T_1$ of $S(\overline{C})$ which induces $\phi_1$.

Suppose first that $D \neq \emptyset$. Then we can regard $S(\overline{C})$ as $S(D) \times S(C)$, and have (because $\phi$ is the identity on $E^B$)

$$
\phi(e(A - \overline{C}) \times x \times e(C)) = [e(A - \overline{C}) \times x] \times e(C) \quad \text{for each } x \in E(D),
$$

$e(A - \overline{C})$ and $e(C)$ denoting the unit elements of $E(A - \overline{C})$, $E(C)$. Hence $\phi_1(x \times e(C)) = \pi_{\overline{C}} \phi \pi_{\overline{C}}^{-1}(x \times e(C)) = x \times e(C)$ for each $x \in E(D)$.

Thus, for each measurable subset $X$ of $S(D)$, $T_1(X \times S(C))$ differs from $X \times S(C)$ by a null set. Apply this to the sets $T_1(X_n)$ ($i = 0, \pm 1, \ldots, n = 1, 2, \ldots$) in turn, where $X_1, X_2, \ldots$, forms a separating sequence of measurable sets in $S(D)$; we obtain countably many null sets with union $N$, say. Then $N$ is null, $T_1(N) = N = T_1^{-1}(N)$, and

$$
T_1[(X_n \times S(C)) - N] = (X_n \times S(C)) - N \quad (n = 1, 2, \ldots).
$$

Define a transformation $T_2$ on $S(\overline{C})$ by:

$$
T_2(p) = T_1(p) \text{ if } p \in S(\overline{C}) - N; \quad T_2(p) = p \text{ if } p \in N.
$$

Clearly $T_2$ is another point-automorphism of $S(\overline{C})$ which also induces $\phi_1$; and we now have

$$
T_2(X_n \times S(C)) = X_n \times S(C) \quad (n = 1, 2, \ldots).
$$

As $X_1, X_2, \ldots$, is a separating sequence, it follows that
\[ T_2(p \times S(C)) = p \times S(C) \quad (p \in S(D)), \]

and hence that
\[ T_2(X \times S(C)) = X \times S(C) \quad \text{for every } X \subseteq S(D). \]

Finally we define \( T \) by
\[ T(p \times q) = p \times T_2(q) \quad \text{where } p \in S(A - C) \quad \text{and} \quad q \in C. \]

Then \( T \) is clearly a point-automorphism of \( S \), and it is easily seen that \( \pi_B T = \pi_B \). To show that \( T \) induces \( \phi \), it is enough to prove (by a double application of Lemma 1) that if \( X \subseteq x \subseteq E(A - C) \), \( Y \subseteq y \subseteq E(D) \), \( Z \subseteq z \subseteq E(C) \), then \( T(X \times Y \times Z) \subseteq \phi(x \times y \times z) \), and this can be done by a straightforward calculation.

If \( D = \emptyset \), we define \( T_2 = T_1 \) on \( S(C) = S(C) \), and define \( p \) as before; only minor (simplifying) adjustments are needed in the argument.

**Lemma 6.** Let \( S = \prod_{\alpha \in A} I_\alpha \), \( \alpha \in A \), and let \( B \) be any subset of \( A \) for which \( A - B \) is countable. Suppose \( \phi \) is a set-automorphism of \( S \), and that \( B \) is invariant under \( \phi \), so that \( \phi \) restricted to \( E^B \) induces an automorphism \( \phi' \) of \( E(B) \). Then, given any point-automorphism \( T' \) of \( S(B) \) which induces \( \phi' \), there exists a point-automorphism \( T \) of \( S \) which induces \( \phi \), and which satisfies \( \pi_B T = T' \pi_B \).

Lemma 5 is the special case when \( T' = \text{identity} \), and we reduce the general case to the special one. By Lemma 1, there exists an automorphism \( \psi \) of \( E \) such that
\[ \psi(x \times y) = \phi'(x) \times y, \quad x \in E(B), \quad y \in E(A - B). \]

Then \( \theta = \phi \psi^{-1} \) is also an automorphism of \( E \); and, using the fact that \( \phi' = \pi_B \phi \pi_B^{-1} \), it is easy to see that \( \theta \) is the identity mapping on \( E^B \). By Lemma 5, there exists a point-automorphism \( T^* \) of \( S \) which induces \( \theta \) on \( E \), and which satisfies \( \pi_B T^* = \pi_B \). Define
\[ T(p \times q) = T^*(T'(p) \times q), \quad p \in S(B), \quad q \in S(A - B). \]

Then \( T \) is clearly a point-automorphism of \( S \), and it is a straightforward matter to verify that \( T \) has the desired properties.

**4. Theorem.** Let \( S = \prod_{\alpha \in A} I_\alpha \), \( \alpha \in A \), and let \( \phi \) be a set-automorphism of \( S \). Then there exists a point-automorphism \( T \) of \( S \) which induces \( \phi \).

Consider the family of ordered pairs \((B_\lambda, T_\lambda)\) where (i) \( B_\lambda \) is a subset of \( A \) which is invariant under \( \phi \), (ii) \( T_\lambda \) is a point-automorphism of \( S(B_\lambda) \), and (iii) the automorphisms of \( E(B_\lambda) \) induced by \( \phi \) restricted to \( E^{B_\lambda} \) (i.e., \( \pi_{B_\lambda} \phi \pi_{B_\lambda}^{-1} \)), and by \( T_\lambda \), are the same. Say that \((B_\lambda, T_\lambda) < (B_\mu, T_\mu)\) provided that \( B_\lambda \subseteq B_\mu \) and \( \pi_{B_\lambda} T_\mu = T_\lambda \pi_{B_\lambda} \) on \( S(B_\mu) \).
Here π_{μλ} is used as an abbreviation for π_{B_μ,B_λ}; similarly we shall abbreviate π_{B_λ} to π_λ. The partial ordering so defined is clearly transitive. Further, every linearly ordered subfamily \{(B_μ, T_μ), μ ∈ M\} has an upper bound in the family. To see this, define B' = ∪ B_μ; this is an invariant subset of A (under Φ). Given p ∈ S(B') and α ∈ B', pick any B_μ ∋ α, and let q_α be the αth coordinate of T_μ(π_μ(p)). It is easy to see that q_α is independent of the choice of μ, and we define T'(p) to be the point of S(B') having αth coordinate q_α (α ∈ B'). A straightforward calculation, using Lemma 2, shows that (B', T') is a member of our family, and that (B_μ, T_μ) ≤ (B', T') for each μ ∈ M.

By Zorn's lemma, it follows that there is a maximal member (B, T) of the family (note that the family is not vacuous, from Lemmas 3 and 4). It is enough to prove that B = A, for condition (iii) above then shows that T induces Φ. Suppose not, and pick a ∈ A - B; by Lemma 3 there is a countable set D ⊂ A, invariant under Φ, which contains α. Let B^* = B ∪ D; then B^* is also invariant, and φ^* = π_{B^*}Φπ_{B^*}^{-1} is an automorphism of S^* = S(B^*). We apply Lemma 6 to the product space S^* with invariant subset B, set-automorphism φ^* and point-automorphism T, obtaining a point-automorphism T^* of S^* which induces φ^* and satisfies π_{B^*}T^* = Tπ_{B^*}. But now (B^*, T^*) is a member of the family defined above, and it contradicts the maximality of (B, T), since (B, T) < (B^*, T^*) and B ≠ B^*. This contradiction establishes the theorem.

REFERENCES


University of Manchester, Manchester, England