ON SPACES WHICH ARE NOT OF COUNTABLE CHARACTER

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It is well known that the unit interval $I$ has a countable base and the fixed point property. By considering the maps $g(x) = x^2$ and $h(x) = 1 - x$, one sees that there is no $x \in I$ such that for every continuous map $f: I \to I$, $x \in f(I)$ implies $f(x) = x$.

In Theorem 1, it is shown that if $A$ is a closed, non-null proper subset of a locally connected, compact Hausdorff space $X$ which has a countable base, then there exists a continuous map $f: X \to X$ such that $A \cap f(X)$ is not contained in $A \cap f(A)$. Theorem 2 shows that certain nondegenerate topological spaces $X$ contain proper subsets $M$ such that for every continuous map $f: X \to X$, $M \cap f(X) \subseteq M \cap f(M)$. That is, for each of these spaces $X$ and every continuous map $f: X \to X$, $x \in M \cap f(X)$ implies $f^{-1}(x) \cap M \neq \emptyset$. The corollary is of interest in that, if $X$ satisfies the hypotheses of Theorem 2 and $M$ consists of a single point, then a fixed point of some of the maps $f: X \to X$ is located.

**Theorem 1.** Suppose $X$ is a connected, locally connected, compact Hausdorff space which has a countable base. If $A$ is any non-null, closed, proper subset of $X$, then there exists a continuous map $f: X \to X$ such that $A \cap f(X) \backslash A \cap f(A) \neq \emptyset$.

**Proof.** Since $X$ is compact Hausdorff and has a countable base, $X$ is metrizable. Hence $X$ is arcwise connected. Let $y \in X \backslash A$. Since

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$X$ is normal, there exists a continuous map $h$ such that $h(x) = 0$ for $x \in A$, $h(y) = 1$, and $0 \leq h(x) \leq 1$ for each $x \in X$. Since $X$ is arcwise connected, there is an arc $C$ connecting $y$ and $A$. Now $C$ contains a subarc $C_1$, such that $y \in C_1$ and $C_1 \cap A$ is a single point $x_0$. Then there is a homeomorphism $g$ such that $g([0, 1]) = C_1$, $g(0) = y$, and $g(1) = x_0$. Consider the continuous map $f = gh$. Clearly $f : X \to X$ and $x_0 \in A \cap f(X)$.

But since $x_0 \notin y$ and $f(A) = y$, $x_0 \notin f(A)$. Hence $x_0 \notin A \cap f(A)$; and $f = gh$ is the required map.

In the following let $M$ consist of the set of all points $x \in X$ such that if $x$ is a limit point of $\{y_n\}$ where $U_{y_n} \subset X \setminus x$, then $\{y_n\}$ contains uncountably many distinct points. It may be noted that $X$ does not satisfy the first axiom of countability at points of $M$.

**Theorem 2.** Let $X$ be a connected Hausdorff space which contains a non-null set $M$ such that $M = \overline{M}$, and $M \neq X$. Suppose also that each point of $X \setminus M$ has a countable base. Then for every continuous map $f : X \to X$, $M \cap f(X) \subset M \cap f(M)$.

**Proof.** Let $f$ be a continuous function such that $f$ maps $X$ into $X$. If $M \cap f(X) = \emptyset$, then $M \cap f(X) \subset M \cap f(M)$. On the other hand, suppose $x \in M \cap f(X)$ and $x \notin f(M)$. Now $x$ is a limit point of $X \setminus x$, for otherwise $X$ would not be connected. Since $x \notin f(M)$, $f^{-1}(x) \subset X \setminus M$.

Suppose there exists $z \in f^{-1}(x)$ such that every neighborhood of $z$ intersects $X \setminus f^{-1}(x)$. Since $X \setminus M$ is open and $z \in X \setminus M$, there exists a countable set $\{U_n(z)\}$ of neighborhoods of $z$ such that $\bigcap_{n=1}^{\infty} U_n(z) = z$, and $U_n(z) \subset X \setminus M$ for each $n$. In each $U_n(z)$ there exists a point $u_n$ such that $u_n \in X \setminus f^{-1}(x)$. Now $f(u_n) \subset X \setminus x$ for each $n$; and, by the continuity of $f$, $x$ is a limit point of the set $\bigcup_{n=1}^{\infty} f(u_n)$. But $\bigcup_{n=1}^{\infty} f(u_n)$ does not contain uncountably many distinct points. Thus a contradiction has been reached. Suppose that for every $z \in f^{-1}(x)$, there exists a neighborhood $U(z)$ such that $U(z) \cap \{X \setminus f^{-1}(x)\} = \emptyset$. Clearly $U(z)$ may be taken so that $U(z) \subset X \setminus M$. Then $U(z) \subset f^{-1}(x)$ for every $z \in f^{-1}(x)$, and $f^{-1}(x)$ is open in $X$. Since $x$ is closed in $X$, $f^{-1}(x)$ is closed in $X$. Therefore $f^{-1}(x) = X$ and $f(X) = x$. But $f(M) \subset f(X)$; hence, $f(M) = x$. But this contradicts the assumption that $x \notin f(M)$.

**Corollary.** If $X$ is a nondegenerate connected Hausdorff space in which $M$ is a single point $x_0$, then for every continuous function $f$ such that $f : X \to X$ and $x_0 \in f(X)$, $f(x_0) = x_0$.

**Proof.** By Theorem 2, $x_0 \in f(x_0)$. Since $f(x_0)$ is a single point, $x_0 = f(x_0)$.

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