THE COHOMOLOGY ALGEBRA OF CERTAIN LOOP SPACES

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The purpose of this paper\textsuperscript{1} is to determine the cohomology algebra of a loop space over a topological space whose cohomology algebra is a truncated polynomial algebra generated by an element of even degree. As special cases we obtain the well-known results when the base space has as cohomology algebra an exterior algebra (the base space an even dimensional sphere) or a polynomial algebra (the base space infinite dimensional complex projective space; compare also Theorem 2 in [1]). In particular, the result is applicable to loop spaces over complex and quaternionic projective $n$-spaces and the Cayley plane.

Throughout, $A$ will denote a commutative ring with unit and $A$-algebra will mean an associative $A$-algebra with unit.

1. Augmented spectral sequences of algebras. A differential $A$-module consists of an $A$-module $E$ and a (module) endomorphism $d: E \to E$ such that $dd = 0$. The map $d$ is called a differential and the elements of its kernel and image are called cycles and boundaries respectively; the quotient module $H(E) = \text{Kernel of } d / \text{Image of } d$ is called the derived module. A differential $A$-algebra consists of an $A$-algebra which is a differential $A$-module and an automorphism $\omega: E \to E$ such that

\begin{equation}
(1.1) \quad d\omega + \omega d = 0, \quad d(xy) = (dx)y + \omega(x)dy, \quad x, y \in E.
\end{equation}

It follows that $H(E)$ has a naturally induced multiplication under which $H(E)$ is an $A$-algebra. An augmentation of a differential $A$-algebra is an algebra homomorphism $\alpha: E \to A$ with right inverse $\beta: A \to E$ such that $\alpha d = 0$. It follows that $H(E)$ has a naturally induced augmentation $\alpha$. The kernel of $\alpha$ will be denoted by $E^+$. An augmented spectral sequence of $A$-algebras is a sequence of augmented differential $A$-algebras $(E_r)$, $r \geq 0$, such that $E_{r+1} = H(E_r)$ and $\alpha_{r+1} = \alpha_r$. The limit of $(E_r)$ is the augmented $A$-algebra defined as follows: An element $x_r \in E_r$ is called a permanent cycle if it is a cycle and its successive projections in $E_{r+1}, E_{r+2}, \cdots$ are cycles. Let $E_\infty$ be the set of sequences $(x_r)$ where $x_r$ is a permanent cycle of $E_r$ and $x_{r+1}$ is the projection of $x_r$ in $E_{r+1}$, with two such sequences identified if $x_r = x'_r$ for all $r \geq r_0$. Defining

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where $\alpha_r$ is the augmentation of $E_r$, makes $E_\infty$ into an augmented $A$-algebra. The augmentation $\alpha$ of $E_\infty$ is well-defined and its kernel $E_*^+$ is the subalgebra defined by permanent cycles $(x_r)$ such that $x_r \in E_r^+$. The spectral sequence is acyclic if $E_\infty^+ = 0$.

An augmented spectral sequence of $A$-algebras is canonical if the sequence $(E_r)$ is defined for $r \geq 2$ and for each $r$:

(a) $E_r$ is a bigraded algebra, $E_r = \sum_{p,q} E_r^{p,q}$, with $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$; moreover, the multiplication in $E_r$ is anticommutative with respect to total degree $p + q$.

(b) The differential $d_r$ is bihomogeneous of bidegree $(r, 1-r)$.

(c) The automorphism $\omega_r$ is given by $\omega_r(x) = (-1)^{p+q} x$ for $x \in E_r^{p,q}$.

(d) The augmentation $\alpha_r$ maps $E_r^{0,0}$ isomorphically onto $A$.

(e) $E_r^{p,q} = H(E_r^{p,q})$.

It follows from (d) that $E_*^+ = \sum_{p+q > 0} E_r^{p,q}$. From (e) it follows that $E_\infty$ has a naturally induced bigrading with $E_\infty^{p,q} = 0$ if $p < 0$ or $q < 0$. In view of (d) it then follows that $\alpha$ maps $E_\infty^{0,0}$ isomorphically onto $A$, and $E_\infty^+ = \sum_{p+q > 0} E_\infty^{p,q}$. Thus acyclicity of the spectral sequence is equivalent to the statement that $E_\infty^{p,q} = 0$ for $p + q > 0$. It may be readily proved that

\begin{equation}
E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q} \quad \text{if } r > p \text{ and } r > q + 1.
\end{equation}

The spectral sequence is said to be initially decomposable if

\begin{equation}
E_2^{p,q} = E_2^{p,0} \cdot E_2^{0,q};
\end{equation}

more precisely, if every $y \in E_2^{p,q}$ can be written as a sum of products $xz$ where $x \in E_2^{p,0}$ and $z \in E_2^{0,q}$. Note that $B = \sum E_2^{p,0}$ and $F = \sum E_2^{0,q}$ are graded subalgebras of $E_2$.

2. Monogenic twisted polynomial algebras. A monogenic twisted polynomial $A$-algebra of height $h$, $2 \leq h \leq \infty$, and type $t = (t_{m,n})$ is a free $A$-module generated by a sequence of elements $x_0, x_1, \ldots, x_{h-1}$ with multiplication defined by

\begin{equation}
x_m x_n = \begin{cases} t_{m,n} x_{m+n} & \text{if } m + n < h, \\ 0 & \text{if } m + n \geq h,
\end{cases}
\end{equation}

where the $t_{m,n}$ are nonzero elements of $A$ which satisfy:

\begin{align*}
t_{0,n} &= 1, & t_{0,0} &= 1, \\
t_{m,n} &= t_{n,m},
\end{align*}
From (2.3) and (2.4) follow commutativity and associativity respectively; from (2.1) and (2.2) follows that $x_0 = 1$. The powers $x_i^m$ are related to the generators $x_m$ as follows: Putting $t_k = t_{1, k-1}$, $(k > 0)$, then by induction one proves

$$x_1^m = t_1 t_2 \cdots t_m x_m.$$  \hspace{1cm} (2.5) 

We shall write $x_1 = x$ and denote the algebra by $A[x, h, t]$. In particular, if $t_{m,n} = 1$ for all $m + n < h$ then the algebra is the ordinary (truncated) polynomial algebra of height $h$ which we shall denote by $A[x, h]$; evidently $A[x, 2]$ is the exterior algebra $\Lambda_A(x)$. If each $t_{m,n}$ differs from the binomial coefficient $(m, n) = m + n! / m! n!$ by a unit then the algebra will be said to be of binomial type.

A monogenic twisted $A$-algebra of binomial type is a free $A$-module generated by a sequence of elements $(x_0, x_1, x_2, \cdots)$ with multiplication defined by

$$x_m x_n = (m, n) x_{m+n}.$$  \hspace{1cm} (2.6) 

It will be denoted by $T_A(x_0, x_1, x_2, \cdots)$. Since the binomial coefficients satisfy (2.2), (2.3), and (2.4), $T_A(x_0, x_1, x_2, \cdots)$ is associative, commutative, and $x_0 = 1$.

We note the following readily proved property of the binomial coefficients $(m, n)$ modulo a prime $p$:

$$(m, n) = (m_0, n_0)(m_1, n_1) \cdots (m_j, n_j),$$  \hspace{1cm} (2.7) 

where

$$m = m_0 + m_1 p + \cdots + m_j p^j, \quad n = n_0 + n_1 p + \cdots + n_j p^j,$$  \hspace{1cm} (2.8) 

are the $p$-adic expansions of $m$ and $n$, and $m_k = 0$ if $k > i$.

**Proposition 1.** (a) If $A$ has characteristic zero then

$$T_A(x_0, x_1, x_2, \cdots) = A[x, \infty, t].$$  \hspace{1cm} (2.9) 

(b) If $A$ has characteristic prime $p$ then there is an algebra isomorphism\textsuperscript{2}

$$\phi: T_A(x_0, x_1, x_2, \cdots) \cong \otimes_{i \geq 0} A[x^p, p, t^{(i)}], \quad t^{(i)}_{m,n} = (m, n).$$  \hspace{1cm} (2.10) 

**Proof.** To prove (a) we need only note that $(m, n) \not= 0$. (b) For each $i \geq 0$ define $p$ elements $y_m^{(i)} = x_m p^i, 0 \leq m < p$. If $0 \leq m < p$ and $0 \leq n < p$ then, using (2.6) and (2.7), we have

\textsuperscript{2} By $\otimes_{i \geq 0}$ is meant the "weak" tensor product.
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\[ y_m^{(i)} y_n^{(i)} = x_{mp^i} x_{np^i} = (m p^i, n p^i) x_{(m+n)p^i} = (m, n) x_{(m+n)p^i}. \]

It is clear that \((m, n) \equiv 0\) if and only if \(m + n \geq p\); hence

\[
y_m^{(i)} y_n^{(i)} = \begin{cases} 
(m, n) y_{m+n}^{(i)} & m + n < p, \\
0 & m + n \geq p.
\end{cases}
\]

Thus (2.1) is satisfied; the remaining conditions (2.3), (2.4), and (2.5) are also satisfied as noted previously. Thus for each \(i\) we have a subalgebra \(A[y^{(i)}, p, t^{(i)}]\) (of binomial type). It remains to show that \(T_A(x_0, x_1, x_2, \ldots)\) is isomorphic to their tensor product.

It follows from (2.6) and (2.7) that corresponding to the \(p\)-adic expansion (2.8) for \(m\), we may write \(x_m\) as a unique product

\[ x_m = x_{m_0} x_{m_1 p} \cdots x_{m_ip^i} = y_{m_0}^{(0)} y_{m_1}^{(1)} \cdots y_{m_i}^{(i)}. \]

Therefore the correspondence defined by

\[ \phi(x_m) = y_{m_0}^{(0)} \otimes y_{m_1}^{(1)} \otimes \cdots \otimes y_{m_i}^{(i)} \]

establishes a module isomorphism (2.10). It remains to show that \(\phi\) is multiplicative. Let (2.8) be the \(p\)-adic expansions of \(m\) and \(n\) (we may assume \(i \leq j\)). Then, using (2.6) and (2.7),

\[
\phi(x_m) \phi(x_n) = (y_{m_0}^{(0)} \otimes \cdots \otimes y_{m_i}^{(i)})(y_{n_0}^{(0)} \otimes \cdots \otimes y_{n_j}^{(j)}) \\
= (m_0, n_0) \cdots (m_j, n_j) y_{m_0+n_0}^{(0)} \otimes \cdots \otimes y_{m_j+n_j}^{(j)} \\
= (m, n) y_{m_0+n_0}^{(0)} \otimes \cdots \otimes y_{m_j+n_j}^{(j)}.
\]

We consider 2 cases:

(i) If \((m, n) \equiv 0 \pmod{p}\) then \(\phi(x_m) \phi(x_n) = 0.\) But also \(\phi(x_m x_n) = \phi(0) = 0.\)

(ii) If \((m, n) \not\equiv 0 \pmod{p}\) then none of the factors \((m_r, n_r), 0 \leq r \leq j,\) are zero and hence \(m_r + n_r < p\) for all \(r.\) But then

\[ m + n = (m_0 + n_0) + (m_1 + n_1) p + \cdots + (m_j + n_j) p^i, \]

\[ \phi(x_{m+n}) = y_{m_0+n_0}^{(0)} \otimes \cdots \otimes y_{m_j+n_j}^{(j)}. \]

Thus (2.11) becomes

\[ \phi(x_m) \phi(x_n) = (m, n) \phi = \phi(x_{m+n}), \]

completing the proof of Proposition 1.

3. The main theorem.

**Theorem 1.** Let \((E_r)\) be an initially decomposable acyclic canonical spectral sequence of \(A\)-algebras. If \(B\) is a truncated polynomial algebra
where $x$ has even degree $m \geq 2$, then

(3.1) \[ F \cong \bigwedge_A (x_1) \otimes T_A(z_0, z_2, z_4, \cdots) \]

where $z_1 \in F$ and is of degree $m-1$ and $z_2 \in F$ and has degree $i(hm-2)$.

In view of Proposition 1 we have:

**Corollary.** (a) If in addition $A$ has characteristic zero then

(3.2) \[ F \cong \bigwedge_A (z_1) \otimes A[z_2, \infty, t], \]

the second factor being of binomial type.

(b) If in addition $A$ has prime characteristic $p$ then

(3.3) \[ F \cong \bigwedge_A (z_1) \otimes \bigoplus_{i \geq 0} A[z_{2p^i}, p, t^{(i)}], \]

each of the monogenic factors in the tensor product being of binomial type.

**Proof.** The following is a trivial consequence of (1.5):

(3.4) \[ E_r^{p \cdot q} = 0 \quad \text{if} \quad E_2^{p \cdot 0} = 0 \quad \text{or} \quad E_2^{0 \cdot q} = 0 \quad (r \geq 2). \]

We note further that the assumption on $B$ gives

(3.5) \[ E_2^{p \cdot 0} = 0, \quad \text{if} \quad p \neq tm, (t = 0, 1, \cdots, h - 1), \]

(3.6) \[ E_2^{t \cdot m \cdot 0} = \begin{cases} 0 & \text{if} \quad t \geq h, \\ A \cdot x^t & \text{if} \quad 0 \leq t < h, \end{cases} \]

(by $A \cdot x^t$ we mean the free $A$-module generated by $x^t$). We shall now prove:

(3.7) \[ E_2 = E_3 = \cdots = E_m, \]

(3.7)\' \[ E_{tm+1} = E_{tm+2} = \cdots = E_{(t+1)m}, \quad t \geq 1, \]

(3.7)\'' \[ E_{(h-1)m+1} = E_{(h-1)m+2} = \cdots = E_{\infty}. \]

If $p \neq sm$ then, in view of (3.4) and (3.6), $E_r^{p \cdot q} = 0$ and hence $d_r(E_r^{p \cdot q}) = 0$. On the other hand, if $p = sm$ but $r$ is not a multiple of $m$ then

\[ d_r(E_r^{sm \cdot q}) \subset E_r^{sm+r \cdot q-r+1} = 0 \]

(the latter module is zero by (3.4) and (3.6) since $sm+r$ is not a multiple of $m$). Thus $d_r = 0$ if $r$ is not a multiple of $m$ and (3.7) and (3.7)\' follow. If $r \geq hm$ then by (3.4) and (3.6), $E_r^{p \cdot r \cdot q-r+1} = 0$ and hence $d_r(E_r^{p \cdot q}) = 0$. Thus $E_{hm} = E_{hm+1} = \cdots = E_{\infty}$. Combining this with (3.7)\' (taking $t = h - 1$) gives (3.7)\''.
**Remark.** Using these results it will be possible to identify $E_{0,q}^0 = E_{0,q}^r$ for some values of $q$ and $r$ ($r > 2$). When we write $d_r(u)$, where $u \in E_{2,q}^0$, such an identification will be implied.

For convenience put $q_0 = hm - 2$. We shall now prove:

A. If $q \not\equiv 0, m - 1$, modulo $q_0$ then $E_{2,q}^0 = 0$.

B. There exists a sequence of generators $z_0 = 1, z_1, z_2, \ldots, z_j, \ldots$ for $F$ such that

\begin{align*}
(3.8) \quad E_{2,q}^{0,jq_0+(m-1)} &= A \cdot z_{2j+1}, & d_m(z_{2j+1}) &= x z_{2j}, \\
(3.9) \quad E_{2,q}^{0,jq_0} &= A \cdot z_{2j}, & d_{(h-1)m}(z_{2j}) &= x^{h-1} z_{2j-1},
\end{align*}

(note the preceding remark).

The proof is by induction on $q$. Let $q > 0$ and assume:

A. Statement A holds for all $q$ such that $0 \leq q < q_0$.

B. We have chosen generators $z_0 = 1, z_{2j}$ ($0 < j q_0 < q$), and $z_{2j+1}$ ($m - 1 \leq j q_0 + m - 1 < q$) such that (3.8) and (3.9) hold.

Clearly A_1 and B_1 are trivial; it remains to prove A_{q+1} and B_{q+1}.

We shall first prove

\begin{equation}
E_{tm+1}^{0,q} = E_{tm}^{0,q}
\end{equation}

holds in the following cases:

(i) $q \equiv 0 \pmod{q_0}$, $1 \leq t < h - 1$.

(ii) $q \equiv m - 1 \pmod{q_0}$, $1 < t \leq h - 1$.

(iii) $q \not\equiv 0$, $m - 1 \pmod{q_0}$, $1 \leq t \leq h - 1$.

Consider

\begin{equation}
\begin{array}{c}
0 \rightarrow E_{tm}^{0,q} \rightarrow E_{tm}^{0,q} \rightarrow E_{tm}^{0,q-tm+1} \rightarrow 0
\end{array}
\end{equation}

Since $d_{tm}d_{tm} = 0$, to prove (3.10) it suffices to show that the last module in (3.11) is zero. If $q - tm + 1 < 0$ this is trivial; hence assume $q - tm + 1 \geq 0$. We may write $q = jq_0 + s$, $0 \leq s < q_0$.

(i) If $s = 0$ and $1 \leq t < h - 1$ then $q - tm + 1 = jq_0 - tm + 1 \not\equiv 0$, $m - 1 \pmod{q_0}$. Therefore it follows from A_q and (3.4) that $E_{tm}^{tm,q-tm+1} = 0$.

(ii) $s = m - 1$ and $1 < t \leq h - 1$ then $q - tm + 1 = jq_0 + (1-t)m \not\equiv 0$, $m - 1 \pmod{q_0}$. As in the preceding case we may conclude that $E_{tm}^{tm,q-tm+1} = 0$.

(iii) Suppose $s \not\equiv 0$, $m - 1$ and $1 \leq t \leq h - 1$. We have $q - tm + 1 \equiv s - tm + 1 \pmod{q_0}$. If further $s \not\equiv tm - 1$, $(t+1)m - 2$, then $q - tm + 1 \not\equiv 0$, $m - 1 \pmod{q_0}$, and as in the preceding two cases we may conclude that the last module in (3.11) vanishes. It remains to consider the two exceptional cases:

If $s = tm - 1$, then $q - tm + 1 = jq_0$. Consider the map
Using (3.7) and (1.4) we have

\[ (3.13) \quad E_m^{t, jq_0} = E_2^{t, jq_0} = E_2^{t, 0, jq_0}. \]

By hypothesis \( E_2^{t, 0} = A \cdot x^t \), and by \( B_{q} \), \( E_2^{0, jq_0} = A \cdot z_{2j} \). It follows from (3.13) that \( E_m^{t, jq_0} = A \cdot x^t z_{2j} \). Similarly we may show \( E_m^{t-1, jq_0} = A \cdot x^{t-1} z_{2j+1} \). Now using the latter part of (3.8) we have

\[ d_m(x^{t-1} z_{2j+1}) = d_m(x^{t-1}) z_{2j+1} + x^{t-1} d_m(z_{2j+1}) = x^t z_{2j}. \]

Thus (3.12) is an isomorphism and \( E_m^{t, jq_0} = 0 \). It follows that the last module in (3.11) is zero.

If \( s = (t + 1)m - 2 \), then \( q - tm + 1 = jq_0 + m - 1 \). Since (3.12) is an isomorphism we have that \( E_m^{t, jq_0} = 0 \). The last module in (3.11) is therefore zero. This completes the proof of (3.10).

Proof of \( A_{q+1} \). Let \( q \neq 0, m - 1 (\mod q_0) \). Using (3.7), (3.7)', (3.7)'', and (3.10) (case iii), we may write \( E_2^{0, \tilde{q}} = E_2^{0, \tilde{q}} \) which is zero by acyclicity.

Proof of \( B_{q+1} \). We assert that the following maps are isomorphisms:

\[ (3.14) \quad d_m: E_m^{0, \tilde{q}} \rightarrow E_m^{m, \tilde{q} - m + 1}, \quad \text{if } \tilde{q} = jq_0 + m - 1; \]

\[ (3.15) \quad d_{(h-1)m}: E_{(h-1)m}^{0, \tilde{q}} \rightarrow E_{(h-1)m}^{(h-1)m, \tilde{q} - (h-1)m + 1}, \quad \text{if } \tilde{q} = jq_0. \]

Assuming this we may prove \( B_{q+1} \) as follows: Let \( \tilde{q} = jq_0 + m - 1 \; : \; \tilde{q} - m + 1 = jq_0 < \tilde{q} \). By (3.9), therefore \( E_2^{0, jq_0} = A \cdot z_{2j} \). Since also \( E_2^{0, 0} = A \cdot x \), it follows from (1.4) that \( E_2^{0, jq_0} = A \cdot x z_{2j} \). Using (3.7) we may replace the subscript \( m \) by \( 2 \) in each module in (3.14). If we therefore define \( z_{2j+1} = d_m(x z_{2j}) \) then (3.8) holds for \( B_{q+1} \). Now let \( \tilde{q} = jq_0 \). Then

\[ \tilde{q} - (h - 1)m + 1 = jq_0 - (h - 1)m + 1 = (j - 1)q_0 + m - 1. \]

Since also \( \tilde{q} - (h - 1)m + 1 < \tilde{q} \), we have by (3.8) \( E_2^{0, \tilde{q} - (h-1)m + 1} = A \cdot z_{2j-1} \). By hypothesis, \( E_2^{0, \tilde{q} - (h-1)m + 1} = A \cdot x^{h-1} \); hence it follows from (1.4) that

\[ (3.16) \quad E_2^{(h-1)m, \tilde{q} - (h-1)m + 1} = A \cdot x^{h-1} z_{2j - 1}. \]

We assert that we may identify

\[ (3.17) \quad E_2^{(h-1)m, \tilde{q} - (h-1)m + 1} = E_2^{(h-1)m, \tilde{q} - (h-1)m + 1}. \]

To prove this let \( 1 \leq t < h - 1 \) and consider

\[ E_2^{(h-1-t)m, \tilde{q} - (h-1)m + 1} \rightarrow E_2^{(h-1)m, \tilde{q} - (h-1)m + 1} \rightarrow E_2^{(h-1-t)m, \tilde{q} - (h-1)m + 1}. \]
where $s$ and $S$ are the appropriate integers. The last module is evidently zero (by 3.4) since $h - 1 + t \geq h$. Also

$$s = \bar{q} - (h - 1)m + 1 + tm - 1 = (j - 1)q_0 + (t + 1)m - 2.$$ 

Evidently $0 < (t + 1)m - 2 < q_0$ and hence $s < \bar{q}$. Moreover, it is readily checked that $(t + 1)m - 2 \neq 0$, $m - 1$. The first module in (3.18) is therefore zero in view of $A_{\bar{q}}$ and (3.4). It follows that

$$E_{(h-1)m, \bar{q}-(h-1)m+1}^{(h-1)m, \bar{q}-(h-1)m+1}, \quad 1 \leq t < h - 1.$$ 

Using (3.7), (3.7)', and (3.19), the identification (3.17) then follows.

We may also identify

$$E_2^{0, jq_0} = E_{(h-1)m}^{0, jq_0}$$

using (3.7), (3.7)', and (3.10) (case i). If we apply (3.17) and (3.20) in (3.15) and define $z_{2j} = d_m^{-1}(x_{h-1}x_{j-1})$ we see that (3.9) holds for $B_{\bar{q}+1}$.

It remains to prove that (3.14) and (3.15) are isomorphisms. Let $\bar{q} = jq_0 + m - 1$ and consider the sequence

$$0 \rightarrow E_{(h-1)m}^{0, \bar{q}} \rightarrow E_{m, jq_0}^{m, \bar{q}} \rightarrow E_{m, jq_0-m+1}^{m, jq_0-m+1}.$$ 

The last module is evidently zero since 2$(h-1)m > hm$. Moreover, $E_{(h-1)m}^{0, jq_0} = 0$, $E_{(h-1)m}^{(h-1)m, jq_0+m-1} = 0$ by (3.7)'' and acyclicity. Thus (3.23) reduces to the isomorphism (3.15). This completes the induction and A and B are proved.

Using A and B we shall now prove the following multiplication relations:
(3.24)  
\[ z_1^2 = 0, \]
(3.25)  
\[ z_1z_{2j} = z_{2j+1}, \]
(3.26)  
\[ z_2z_{2j} = (i, j)z_{2(i+j)}. \]

Note that \( z_1^2 \) has degree \( 2m - 2 \) and \( z_2 \) has degree \( hm - 2 \). Thus if \( h > 2, z_1^2 = 0 \). If \( h = 2 \) put \( z_1^2 = t \), \( t \in A \); then by (3.8) and (3.9) respectively, we have

\[
d_m(z_1^2) = xz_1 - z_1x = xz_1 - xz_1 = 0, \quad d_m(z_1) = txz_1. \]

Since \( xz_1 \) generates a free \( A \)-module, \( t = 0 \) and (3.24) is established.

For \( j = 0 \), (3.24) is trivial. Let \( j > 0 \) and assume (3.25) for all \( j' < j \). Put \( z_1z_{2j} = tz_{2j+1} \), \( t \in A \); then using (3.8) we have

\[
(3.27) \quad d_m(z_1z_{2j}) = xz_{2j} - z_1d_m(z_{2j}). \]

If \( h = 2 \) then using (3.9), the inductive assumption, and (3.24) in succession,

\[
z_1d_m(z_{2j}) = z_1(xz_{2j-1}) = xz_1z_{2j-2} = 0. \]

If \( h > 2 \), note that \( d_m(z_{2j}) \in E_{2m-1}^{m-1} \) which is zero by (3.4) and A. In either case (3.27) reduces to \( d_m(z_1z_{2j}) = xz_{2j} \). But by (3.8),

\[
d_m(z_1z_{2j}) = td_m(z_{2j+1}) = txz_{2j}. \]

Since \( xz_{2j} \) generates a free \( A \)-module, \( t = 1 \) and (3.25) is proved.

For \( i + j = 0 \), (3.26) is trivial. Let \( i + j > 0 \) and assume (3.26) for all \( i' + j' < i + j \). If we put \( z_2z_{2j} = t_{i, j}z_{2(i+j)} \), then using (3.9),

\[
d_{(h-1)m}(z_{2(i+j)}) = (x^{h-1}z_{2i-1})z_{2j} + z_{2i}(x^{h-1}z_{2j-1}) = x^{h-1}z_1(z_{2i}z_{2j} + z_{2i}z_{2j-2}) = x^{h-1}z_1[(i - 1, j) + (i, j - 1)]z_{2i+2j-2} = (i, j)x^{h-1}z_{2i+2j-1}. \]

But also by (3.9),

\[
d_{(h-1)m}(z_{2(i+j)}) = t_{i, j}d_{(h-1)m}(z_{2(i+j)}) = t_{i, j}x^{h-1}z_{2(i+j)-1}. \]

Since \( x^{h-1}z_{2(i+j)-1} \) generates a free \( A \)-module, \( t_{i, j} = (i, j) \).

The theorem now follows from (3.24), (3.25), and (3.26). For by (3.24), \( z_1 \) spans the subalgebra \( A(z_1) \); by (3.26), the elements \( z_{2j} \) span the subalgebras \( T_A(z_0, z_2, z_4, \ldots) \); and by (3.25), \( F \) is clearly isomorphic to the tensor product of the two subalgebras under the obvious map.

4. Topological applications. Let \( J \) denote a principal ideal domain of characteristic \( p \) (\( p \) is then zero or a prime).
Theorem 2. Let $X$ be a topological space whose singular cohomology algebra $H^*(X, J) = J[x, h]$ where $x$ is an element of even degree $m \geq 2$. Let $\Omega$ denote the loop space of $X$ at a base point $x_0$.

(a) $H^*(\Omega, J) \cong \bigwedge_J (z_1) \otimes J[z_2, \infty, t]$, if $p = 0$, where $z_1$ and $z_2$ have degree $m - 1$ and $hm - 2$, respectively, and the second factor is of binomial type.

(b) $H^*(\Omega, J) \cong \bigwedge_J (z_1) \otimes \mathbb{Z}_p[z_2, p, t^{(i)}]$, if $p$ is prime, where $z_1$ and $z_2p^i$ have degrees $m - 1$ and $p^i(hm - 2)$ respectively, and each $t^{(i)}$ is of binomial type.

Proof. Evidently $X$ is arcwise connected, simply connected, and torsion-free. Associated with the Serre fibering $f: E \to X$ where $E$ is the space of paths beginning at $x_0$ (and $\Omega$ is the fibre at $x_0$) is a canonical spectral sequence of $J$-algebras $(E_r)$ which is acyclic (since $E$ is contractible) and such that $E_2^{pq} \cong H^p(X, J) \otimes H^q(\Omega, J)$ (since $X$ is a torsion-free and $J$ is a principal ideal domain). These isomorphisms give an identification of the bigraded $J$-algebras, $E_2 = H^*(X, J) \otimes H^*(\Omega, J)$. Since the spectral sequence is initially decomposable, Theorem 2 follows immediately from Theorem 1 and its corollary.

Let $P_n$ and $Q_n$ denote the complex and quaternionic projective $n$-spaces, $1 \leq n \leq \infty$, respectively, and let $C$ denote the Cayley plane. Their cohomology algebras are known to be

$H^*(P_n, J) = J[x, n + 1]$, degree of $x = 2$;
$H^*(Q_n, J) = J[x, n + 1]$, degree of $x = 4$;

Thus, Theorem 2 applies to $P_n$, $Q_n$, and $C$.

References


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* See reference [2, Chapter IV].