LACUNARY FOURIER SERIES ON NONCOMMUTATIVE GROUPS

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1. Introduction. In classical Fourier analysis lacunary series play a considerable role due to theorems of Kolmogoroff, Banach, Sidon and others. According to the usual definition a Fourier series $\sum a_k e^{i k x}$ is called lacunary if $n_{k+1}/n_k > \lambda (> 1)$ for all $k$. This definition makes use of the ordering of the integers and does not immediately extend to two dimensions or to more general groups which have been recognized as a proper setting for large parts of Fourier Analysis.

Let $G$ be a compact group and as usual let $\hat{G}$ denote the set of equivalence classes of unitary irreducible representations of $G$. The set $\hat{G}$ has the following “hypergroup” structure: To each pair $\alpha, \beta \in \hat{G}$ there is attached a measure $\mu_{\alpha, \beta}$ on $\hat{G}$. This is determined by the decomposition of the Kronecker product $\alpha \otimes \beta$. In terms of this structure there is a natural duality between normal subgroups of $G$ and certain subhypergroups of $\hat{G}$. Some of the abelian Pontrjagin duality extends to this situation, although two nonisomorphic finite groups $G$ may have the same hypergroup structure of $\hat{G}$.

The purpose of this note is to point out how, in certain instances, the hypergroup structure of $\hat{G}$ is related to properties of Fourier expansions on $G$. In particular we give a definition of a lacunary Fourier series on $G$ in terms of $\hat{G}$. If $G$ is the circle group, our definition is formally quite different from the usual one but has similar implications. The definition is wide enough to cover the case of a series of the form $\sum a_n e^{i x_n}$, where $x_n$ are independent variables and a well known theorem of Kolmogoroff about such series can be extended to Fourier series on the product $\prod_n U(n)$, $U(n)$ denoting the unitary group in $n$ dimensions. Furthermore, the theorem of Banach stating that a lacunary $L^1$-series is an $L^2$-series is generalized to noncommutative groups.

2. The duality. We shall be concerned with compact groups $G$ with normalized Haar measure $dg$ and the familiar function spaces $L^1(G)$ and $L^2(G)$ of integrable and square integrable functions. The corresponding norms are denoted $\| \|$ and $\| \|_2$.

Every function $f \in L^1(G)$ can be uniquely represented by a Fourier series

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(1) \[ f(g) \sim \sum_{x \in \hat{G}} d_x \text{Tr} \left\langle A_x D_x(g) \right\rangle \]

where \( \text{Tr} \) denotes the usual trace, \( \hat{G} \) is the set of equivalence classes of irreducible unitary representations of \( G \), \( D_x \) is a member of the class \( x \), \( d_x \) is the degree of \( x \) and \( A_x \) is the linear transformation determined by

(2) \[ A_x = \int g f(g) D_x(g^{-1}) \, dg. \]

For the expansion above, the Schur-Peter-Weyl formula is valid

(3) \[ \int g |f(g)|^2 \, dg = \sum_{x \in \hat{G}} d_x \text{Tr} \left\langle A_x A_x^* \right\rangle \]

finiteness of one side implying the finiteness of the other. \( (B^*) \) denotes the adjoint of the operator \( B \). In case \( f \) is a central function, that is \( f \) is invariant under inner automorphisms, the Fourier series (1) takes the form

(4) \[ f(g) \sim \sum a_x \chi(x) \]

where \( g \rightarrow \chi(x) \) is the character of the class \( x \).

The series (1) we call absolutely convergent if \( \sum d_x \| A_x \| < \infty \), \( \| \| \) denoting the usual norm.

**Definition 2.1.** A set \( S \) is called a (discrete) hypergroup if there is given a mapping \( (\alpha, \beta) \rightarrow \mu_{\alpha, \beta} \) of \( S \times S \) into the set of measures on \( S \). A subset \( T \) of the hypergroup \( S \) is called a subhypergroup if all the measures \( \mu_{\alpha, \beta} (\alpha, \beta \in T) \) have support contained in \( T \).

If \( G \) is abelian, \( \hat{G} \) is a group, and if \( G \) is nonabelian every tensor product \( \alpha \otimes \beta \) for \( \alpha, \beta \in \hat{G} \) has a direct decomposition into irreducible unitary components. This induces a hypergroup structure in \( \hat{G} \). If \( A \) and \( B \) are two representations of \( G \) we call \( A \) and \( B \) disjoint if no irreducible component of \( A \) is equivalent to an irreducible component of \( B \).

The identity transformation of arbitrary dimension will be called \( E \) and the irreducible unit representation of \( G \) will be called \( \mathbb{I} \). If \( M \) is an arbitrary subset of \( G \) we let \( M^\perp \) stand for the set of classes \( \alpha \in \hat{G} \) such that \( D_\alpha(g) = E \) for each \( g \in M \). Similarly if \( \mathbb{S} \in \hat{G} \) we let \( \mathbb{S}^\perp \) denote the subset of \( G \) determined by the equations \( D_\alpha(g) = E \) for each \( \alpha \in \mathbb{S} \). For simplification we call a subhypergroup \( \mathbb{S} \) of \( \hat{G} \) a normal subhypergroup if \( I \in \mathbb{S} \) and if \( \alpha \in \mathbb{S} \) implies \( \bar{\alpha} \in \mathbb{S} \) (the bar denotes complex conjugation).

We have then the following duality between normal subhypergroups of \( \hat{G} \) and closed normal subgroup of \( G \). This is closely related to a duality outlined by van Kampen [3].
Theorem 1.

(i) If $M \subset G$, $M^\perp$ is a normal subhypergroup of $\hat{G}$ and $(M^\perp)^\perp$ is the smallest closed normal subgroup of $G$ containing $M$.

(ii) If $\mathcal{S} \subset \hat{G}$, $\mathcal{S}^\perp$ is a closed normal subgroup of $G$ and $(\mathcal{S}^\perp)^\perp$ is the smallest normal subhypergroup of $\hat{G}$ containing $\mathcal{S}$.

(iii) If $N$ is a closed normal subgroup of $G$, $(G/N)^\perp = N^\perp$.

Proof. (i) Let $\alpha, \beta \in M^\perp$ and let $D_\alpha, D_\beta$ be corresponding representations. From the direct decomposition of $D_\alpha \otimes D_\beta$ we get for the characters the decomposition

$$a(g)\beta(g) = \chi_1(g) + \cdots + \chi_n(g)$$

where the $\chi_i$ are characters of irreducible representations whose dimensions $d_i$ satisfy

$$d_\alpha \cdot d_\beta = d_1 + \cdots + d_n.$$ 

Now if $g \in M$ we have $a(g) = d_\alpha$ and $\beta(g) = d_\beta$ and since $\max_\mathbb{R} |\chi(g)| = d_\chi$ we conclude from (4) and (5) that $\chi_i(g) = d_i$ for all $i$. It follows, that $D_{\chi_i}(g) = E$ so $\chi_i \in M^{\perp}$ and $M^{\perp}$ is a subhypergroup which is normal. It is obvious that $(M^{\perp})^{\perp}$ is a closed normal subgroup containing $M$. If $N$ is some arbitrary closed normal subgroup containing $M$, then $(M^{\perp})^{\perp} \subset N$ because if $n \in (M^{\perp})^{\perp} - N$ we can (by going to the factor group $G/N$) find a representation $D \in \hat{G}$ such that $D(g) = E$ for $g \in N$ but $D(n) \neq E$. Then $D \in N^{\perp} - M^{\perp}$ which contradicts $M \subset N$.

(ii) If $\mathcal{S} \subset \hat{G}$ it is clear that $\mathcal{S}^\perp$ is a closed normal subgroup and $(\mathcal{S}^\perp)^\perp$ is a normal subhypergroup. The matrix elements from $\mathcal{S}$ and $I$ can be regarded as a family of continuous functions on $G/\mathcal{S}^\perp$ which separates points. Let $\mathfrak{R}$ be the set of linear combinations of matrix elements from the normal subhypergroup $\mathcal{S}^\ast$ generated by $\mathcal{S}$ and $I$. By the Peter-Weyl theorem $\mathfrak{R}$ is uniformly dense in the space of continuous functions on $G/\mathcal{S}^\perp$. Now if there were a representation $D \in \mathcal{S}^{\perp \perp} - \mathcal{S}^\ast$, each matrix element $a(g)$ from $D$ could be uniformly approximated by elements of $\mathfrak{R}$ but on the other hand $a(g)$ is orthogonal to $\mathfrak{R}$ by the orthogonality relations. This shows that $\mathcal{S}^\ast = \mathcal{S}^{\perp \perp}$.

(iii) Proof obvious.

3. Multipliers.

Definition 3.1. A hyperfunction on $\hat{G}$ is a mapping which assigns to each $\chi \in \hat{G}$ a linear transformation of a complex vector space of dimension $d_\chi$.

Definition 3.2. A hyperfunction $\Gamma$ on $\hat{G}$ is called a multiplier if for each Fourier series for a continuous function

$$f(g) \sim \sum d_\chi \text{Tr} \langle A_\chi D_\chi(g) \rangle$$

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the series
\[
fr(g) \sim \sum dx \operatorname{Tr} (\Gamma_x A_x D_x(g))
\]
is also a Fourier series for a continuous function.

It is easy to see from the closed graph theorem that if \( \Gamma \) is a multiplier there exists a bounded measure \( \mu_\Gamma \) on \( G \) such that \( fr = f \ast \mu_\Gamma \) (convolution product). Less trivial is the following extension of a theorem of Sidon:

**Theorem 2.** Let \( \Gamma \) be a multiplier such that for each Fourier series (6) (with continuous \( f \)) the corresponding series (7) is absolutely convergent. Then there exists a function \( F \in L^2(G) \) such that
\[
fr = f \ast F \quad \text{for all continuous} \ f.
\]

**Proof.** Let \( \Gamma \) be a multiplier with the properties stated in the theorem. Then \( B\Gamma \) is also of that type provided \( B \) is a hyperfunction on \( \hat{G} \) satisfying \( \sup_x \| B_x \| < \infty \). To see that \( B\Gamma \) really is a multiplier we remark that an absolutely convergent series (in the sense defined in this paper) is uniformly convergent on \( G \); this last fact is easily verified by writing each Fourier matrix \( A_x \) as \( P_x V_x \) where \( P_x \) is positive definite and \( V_x \) is unitary. Using a previous remark we see that there exist bounded measures \( \mu_\Gamma \) and \( \mu_{B\Gamma} \) on \( G \) such that
\[
f_{B\Gamma} = f \ast \mu_{B\Gamma},
\]
\[
fr = f \ast \mu_\Gamma.
\]
The hyperfunctions \( B \) satisfying \( \sup_x \| B_x \| < \infty \) (the bounded hyperfunctions) form a Banach space under the norm \( \sup_x \| B_x \| \). The mapping \( T: B \rightarrow \mu_{B\Gamma} \) is a linear mapping of the Banach space of bounded hyperfunctions into the Banach space of measures on \( G \) and again from the closed graph theorem it follows easily that this mapping is continuous. Now an integrable function on \( G \) can be identified with a hyperfunction on \( \hat{G} \) via the Fourier series expansion. If the function \( \phi \in L^1(G) \) corresponds to \( B \) in this manner we see that \( \mu_{B\Gamma} \) is absolutely continuous with respect to Haar measure and has a derivative, say \( \phi^\Gamma \in L^1(G) \). Then the mapping \( \bar{T}: \phi \rightarrow \phi^\Gamma \) is a linear transformation of \( L^1(G) \) into itself and since \( T \) above is continuous it follows that \( \bar{T} \) is spectrally continuous in the sense of [2]. Furthermore \( \phi^\Gamma = \mu_\Gamma \ast \phi \) so \( \bar{T} \) commutes with right translations on \( G \).

This being established, Theorem 2 follows from Theorem \( A \) in [2] which is an extension of a theorem of Littlewood and states that the spectrally continuous operators that commute with right translations are precisely the left convolutions with \( L^2 \)-functions on \( G \).
Definition 3.3. A subset $S \subseteq \hat{G}$ is called distinguished if for every Fourier series for a continuous function
\begin{equation}
    f(g) \sim \sum_{x \in G} d_x \operatorname{Tr} \langle A_x D_x(g) \rangle
\end{equation}

the subseries
\begin{equation}
    \sum_{x \in S} d_x \operatorname{Tr} \langle A_x D_x(g) \rangle
\end{equation}
also represents a continuous function $f_S$.

For abelian groups $G$, the distinguished sets were investigated in [1]. For the noncommutative case a partial description is given by

Theorem 3. The distinguished sets that preserve positivity in the sense that $f_S \geq 0$ whenever $f \geq 0$ are precisely the normal subhypergroups of $\hat{G}$.

Proof. The mapping $f \mapsto f_S$ is continuous (uniform topology) by the closed graph theorem and commutes with left translations. Hence there exists a bounded measure $\mu_S$ on $G$ such that $f_S = f \circ \mu_S$ for all $f$. The mapping $f \mapsto f_S$ also commutes with right translations, so that the Fourier-Stieltjes series for $\mu_S$ has the form

\begin{equation}
    \mu_S(g) \sim \sum_{x \in S} d_x \chi(g).
\end{equation}

Using the assumption of the theorem we see that $\mu_S$ is a positive measure so by a theorem of Wendel [7] the condition $\mu_S \ast \mu_S = \mu_S$ implies that there exists a compact subgroup $K$ of $G$ such that $\mu_S(A) = \mu(A \cap K)$ for every Borel set $A$, where $\mu$ is the Haar measure on $K$. From the Fourier-Stieltjes series for $\mu_S$ we see that

\begin{equation}
    \int_G \overline{D}_x(g) d\mu_S(g) = \int_K \overline{D}_x(k) d\mu(k) = \begin{cases} E & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases}
\end{equation}

which implies $\overline{D}_x(k) = E$ for all $k$ if and only if $x \in S$ or otherwise expressed: $K^\perp = S$.

On the other hand, if $S$ is a normal subhypergroup then the Haar measure $\mu$ on $S^\perp$ extended to a measure on $G$ by $\mu(E) = \mu(E \cap S^\perp)$ has the Fourier-Stieltjes series

\begin{equation}
    \mu(g) \sim \sum_{x \in S} d_x \chi(g)
\end{equation}

and for each continuous function $f$ on $G$ with Fourier series (8) the continuous function $f \ast \mu$ has Fourier series (9), proving that $S$ is distinguished.
4. Lacunary series. In this section we discuss extensions of theorems of Kolmogoroff and Banach.

Let $I$ be a set and to each element $i$ of $I$ attached an integer $d_i$. We consider the compact group $G = \prod_{i \in I} U(d_i)$ where $U(m)$ denotes the unitary group in $m$ dimensions. The projection $D_i$ of $G$ onto $U(d_i)$ is a unitary representation which clearly is irreducible, in other words $I$ can be regarded as a subset of $\widehat{G}$. We consider Fourier series of the form

$$\sum_{i \in I} d_i \text{Tr} \langle A_i D_i(g) \rangle$$

and we shall now indicate the proof of the following theorem.

**Theorem 4.** Suppose $f \in L^1(G)$ and has a Fourier series of the form (10). Then $f \in L^2(G)$ and moreover $2^{-1/2} \|f\|_2 \leq \|f\|_1 \leq \|f\|_2$.

In the case where $d_i = 1$ for each $i \in I$, this result is a well known theorem of Kolmogoroff.

The essence of Theorem 4 is proved in [2]. In fact let us consider a finite subset $J$ of $I$ and a series of the form

$$s(g) = \sum_{j \in J} d_j \text{Tr} \langle B_j D_j(g) \rangle.$$ 

It is then clear that

$$\int_G |s(g)|^m \, dg = \int_{V_J} \left| \sum_j d_j \text{Tr} \langle B_j D_j \rangle \right|^m \, dD_J$$

where $dD_J$ denotes the Haar measure on the product $V_J = \prod_{j \in J} U(d_j)$. By the proof of Lemma 4.1 in [2] and the relation (4.12) in [2] we get

$$\int_G |s(g)|^4 \, dg \leq 2 \left[ \int_G |s(g)|^2 \, dg \right]^2.$$ 

Using the inequality

$$\left[ \int |h(g)| \, dg \right]^2 \geq \left[ \int |h(g)|^4 \, dg \right]^{-1} \cdot \left[ \int |h(g)|^2 \, dg \right]^2,$$

which is a special case of Hölder's inequality we obtain

$$\int_G |s(g)| \, dg \geq 2^{-1/2} \left[ \int_G |s(g)|^2 \, dg \right]^{1/2}.$$ 

By standard approximation arguments the function $f$ can be ap-
proximated in $L^1$-norm by functions of the form (11), and (13) becomes valid for the function $f$. Theorem 4 is proved.

We shall now see that the set $I$ in the above situation appears as a lacunary subset of $\widehat{G}$ in a certain sense.

Let again $G$ be an arbitrary compact group. If $\alpha, \beta \in \widehat{G}$ we denote by $D_\alpha$ and $D_\beta$ arbitrary members of $\alpha$ and $\beta$ respectively, $d_\alpha$ and $d_\beta$ the corresponding dimensions and $n_{\alpha,\beta}$ the number of irreducible components in $D_\alpha \otimes D_\beta$ (counted with multiplicity).

**Definition 4.1.** A subset $S \subset \widehat{G}$ is called lacunary if the two following conditions are satisfied.

(I) Whenever $(\alpha, \beta)$ and $(\gamma, \delta)$ are different pairs from $S$ (that is, the characters $\alpha + \beta$ and $\gamma + \delta$ are different) $D_\alpha \otimes D_\beta$ and $D_\gamma \otimes D_\delta$ are disjoint.

(II) There exists a constant $K$ such that $n_{\alpha,\beta} < K$ for all $\alpha, \beta \in S$.

A Fourier series of the form $\sum_{x \in S} d_x \text{Tr} \langle A_x D_x(g) \rangle$ is called lacunary if $S$ is lacunary.

The following statements show that the series (10) is indeed lacunary.

(i) $D_i \otimes D_j$ is irreducible if $i \neq j$.

(ii) If $d_i = 1$ then $D_i \otimes D_i$ is irreducible.

(iii) If $d_i \geq 2$ then $D_i \otimes D_i$ decomposes into two irreducible parts (of dimensions $(d_i^2 + d_i)/2$ and $(d_i^2 - d_i)/2$).

(iv) $D_i \otimes D_i$ is disjoint from $D_j \otimes D_j$ if $i \neq j$.

(i) and (ii) are obvious. (iii) is a corollary of Lemma 4.1 in [2] combined with the fact that the space of symmetric and antisymmetric tensors are left invariant by $D_i \otimes D_i$. Concerning (iv) we remark that the number of irreducible components common to $D_i \otimes D_i$ and $D_j \otimes D_j$ is equal to

$$\int_G (x_i^2 x_i^2) dg = \int_{U(d_m) \times U(d_i)} (\text{Tr} D_i)^2 (\text{Tr} D_j^{-1})^2 dD_i dD_j.$$ 

($dD_m$ is the Haar measure on $U(d_m)$), and this last integral vanishes as shown in the proof of the cited lemma.

For a general compact group we have a simple result in similar direction.

**Theorem 5.** Let $f$ be a central function in $L^1(G)$ and suppose $f$ has a lacunary Fourier series. Then $f \in L^2(G)$.

**Proof.** The Fourier expansion of $f$ can be written $f(g) \sim \sum_{x \in S} a_x \chi(g)$ where $a_x$ is a complex number and $S$ is lacunary. We consider a finite partial sum $s(g) = \sum_{i=1}^{N} a_n \chi_n(g)$. Then
If we here expand $\chi_p^2$ and $\chi_p \chi_q$ into a sum of characters the same $\chi_i$ will not occur more than once due to condition (I), and we get easily

$$\int_G |s(g)|^2 \, dg = \sum_{p=1}^{N} |a_p|^2 n_{p,p} + 2 \sum_{p \neq q} |a_p|^2 |a_q|^2 n_{p,q}$$

$$\leq K \left( \sum_{p} |a_p|^2 \right)^2.$$

Using the inequality (12) we obtain the conclusion $f \in L^2(G)$ in exactly the same manner as before.

Concerning the relation with the classical definition of lacunary series we remark that a series of the form $\sum a_k e^{i k x}$ where $n_{k+1}/n_k > 2$ for all $k$, is lacunary in the sense of Definition 4.1. This is easily verified and is indeed a basic property in the proof of most classical theorems on lacunary series.

As a simple consequence of Theorem 5 we mention the following fact:

**Theorem 6.** Let $G$ be an abelian group which is compact and not totally disconnected. Suppose the infinite series

$$\sum a_x \sigma \cdot \chi(g)$$

is a Fourier series for some $L^1$-function for each permutation $\sigma$ of $G$. Then $\sum_{x \in \mathcal{G}} |a_x|^2 < \infty$.

**Proof.** By a theorem of Pontrjagin $\mathcal{G}$ has an infinite cyclic subgroup and therefore an infinite countable lacunary subset. Now we can write

$$\sum_{a_x \neq 0} a_x \chi(g) = \sum_S a_x \chi(g) + \sum_T a_x \chi(g)$$

where both sets $S$ and $T$ are infinite and $\sum_S |a_x| < \infty$. Hence $\sum_T a_x \sigma \cdot \chi(g)$ is an $L^1$-series for every permutation $\sigma$ of $\mathcal{G}$. Choose this permutation in such a way that $\sum_T a_x \sigma \cdot \chi(g)$ is a lacunary series and apply Theorem 5. Q.E.D.

If $G$ is an arbitrary compact group, $\mathcal{G}$ need not possess any infinite lacunary subsets. As a simple example we mention $G = SU(2)$. The hypergroup structure of $\mathcal{G}$ is here described by the Clebsch-Gordan formula [8] which shows that the requirement (II) in Definition 4.1 is not fulfilled for any infinite subset $S \subset \mathcal{G}$. 

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