LACUNARY FOURIER SERIES ON NONCOMMUTATIVE GROUPS

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1. Introduction. In classical Fourier analysis lacunary series play a considerable role due to theorems of Kolmogoroff, Banach, Sidon and others. According to the usual definition a Fourier series \( \sum a_k e^{in_k x} \) is called lacunary if \( n_{k+1}/n_k > \lambda (> 1) \) for all \( k \). This definition makes use of the ordering of the integers and does not immediately extend to two dimensions or to more general groups which have been recognized as a proper setting for large parts of Fourier Analysis.

Let \( G \) be a compact group and as usual let \( \hat{G} \) denote the set of equivalence classes of unitary irreducible representations of \( G \). The set \( \hat{G} \) has the following “hypergroup” structure: To each pair \( \alpha, \beta \in \hat{G} \) there is attached a measure \( \mu_{\alpha,\beta} \) on \( \hat{G} \). This is determined by the decomposition of the Kronecker product \( \alpha \otimes \beta \). In terms of this structure there is a natural duality between normal subgroups of \( G \) and certain subhypergroups of \( \hat{G} \). Some of the abelian Pontrjagin duality extends to this situation, although two nonisomorphic finite groups \( G \) may have the same hypergroup structure of \( \hat{G} \).

The purpose of this note is to point out how, in certain instances, the hypergroup structure of \( \hat{G} \) is related to properties of Fourier expansions on \( G \). In particular we give a definition of a lacunary Fourier series on \( G \) in terms of \( \hat{G} \). If \( G \) is the circle group, our definition is formally quite different from the usual one but has similar implications. The definition is wide enough to cover the case of a series of the form \( \sum a_n e^{ix_k} \), where \( x_n \) are independent variables and a well known theorem of Kolmogoroff about such series can be extended to Fourier series on the product \( \prod_n U(n) \), \( U(n) \) denoting the unitary group in \( n \) dimensions. Furthermore, the theorem of Banach stating that a lacunary \( L^1 \)-series is an \( L^2 \)-series is generalized to noncommutative groups.

2. The duality. We shall be concerned with compact groups \( G \) with normalized Haar measure \( dg \) and the familiar function spaces \( L^1(G) \) and \( L^2(G) \) of integrable and square integrable functions. The corresponding norms are denoted \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \). Every function \( f \in L^1(G) \) can be uniquely represented by a Fourier series

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\[ f(g) \sim \sum_{x \in \hat{G}} d_x \text{Tr} \left< A_x D_x(g) \right> \]

where \( \text{Tr} \) denotes the usual trace, \( \hat{G} \) is the set of equivalence classes of irreducible unitary representations of \( G \), \( D_x \) is a member of the class \( x \), \( d_x \) is the degree of \( x \) and \( A_x \) is the linear transformation determined by

\[ A_x = \int_{g} f(g) D_x(g^{-1}) \, dg. \]

For the expansion above, the Schur-Peter-Weyl formula is valid

\[ \int_{g} |f(g)|^2 \, dg = \sum_{x \in \hat{G}} d_x \text{Tr} \left< A_x A_x^* \right> \]

finiteness of one side implying the finiteness of the other. \((B^*)\) denotes the adjoint of the operator \( B \). In case \( f \) is a central function, that is \( f \) is invariant under inner automorphisms, the Fourier series (1) takes the form \( f(g) \sim \sum a_x \chi(g) \) where \( g \rightarrow \chi(g) \) is the character of the class \( x \).

The series (1) we call absolutely convergent if \( \sum d_x^2 \| A_x \| < \infty \), \( \| \| \) denoting the usual norm.

**Definition 2.1.** A set \( S \) is called a (discrete) hypergroup if there is given a mapping \((\alpha, \beta) \rightarrow \mu_{\alpha, \beta}\) of \( S \times S \) into the set of measures on \( S \). A subset \( T \) of the hypergroup \( S \) is called a subhypergroup if all the measures \( \mu_{\alpha, \beta} \) \((\alpha, \beta \in T)\) have support contained in \( T \).

If \( G \) is abelian, \( \hat{G} \) is a group, and if \( G \) is nonabelian every tensor product \( \alpha \otimes \beta \) for \( \alpha, \beta \in \hat{G} \) has a direct decomposition into irreducible unitary components. This induces a hypergroup structure in \( \hat{G} \). If \( A \) and \( B \) are two representations of \( G \) we call \( A \) and \( B \) disjoint if no irreducible component of \( A \) is equivalent to an irreducible component of \( B \).

The identity transformation of arbitrary dimension will be called \( E \) and the irreducible unit representation of \( G \) will be called \( I \). If \( M \) is an arbitrary subset of \( G \) we let \( M^\perp \) stand for the set of classes \( \alpha \in \hat{G} \) such that \( D_\alpha(g) = E \) for each \( g \in M \). Similarly if \( \mathcal{S} \in \hat{G} \) we let \( \mathcal{S}^\perp \) denote the subset of \( G \) determined by the equations \( D_\alpha(g) = E \) for each \( \alpha \in \mathcal{S} \). For simplification we call a subhypergroup \( \mathcal{S} \) of \( \hat{G} \) a normal subhypergroup if \( I \in \mathcal{S} \) and if \( \alpha \in \mathcal{S} \) implies \( \bar{\alpha} \in \mathcal{S} \) (the bar denotes complex conjugation).

We have then the following duality between normal subhypergroups of \( \hat{G} \) and closed normal subgroups of \( G \). This is closely related to a duality outlined by van Kampen [3].
**Theorem 1.**

(i) If \( M \subseteq G \), \( M^\perp \) is a normal subhypergroup of \( \hat{G} \) and \( (M^\perp)^\perp \) is the smallest closed normal subgroup of \( G \) containing \( M \).

(ii) If \( \mathcal{S} \subseteq \hat{G} \), \( \mathcal{S}^\perp \) is a closed normal subgroup of \( G \) and \( (\mathcal{S}^\perp)^\perp \) is the smallest normal subhypergroup of \( \hat{G} \) containing \( \mathcal{S} \).

(iii) If \( N \) is a closed normal subgroup of \( G \), \( (G/N)^\perp = N^\perp \).

**Proof.** (i) Let \( \alpha, \beta \in M^\perp \) and let \( D_\alpha, D_\beta \) be corresponding representations. From the direct decomposition of \( D_\alpha \otimes D_\beta \) we get for the characters the decomposition

\[
\alpha(g)\beta(g) = \chi_1(g) + \cdots + \chi_n(g)
\]

where the \( \chi_i \) are characters of irreducible representations whose dimensions \( d_i \) satisfy

\[
d_\alpha \cdot d_\beta = d_1 + \cdots + d_n.
\]

Now if \( g \in M \) we have \( \alpha(g) = d_\alpha \) and \( \beta(g) = d_\beta \) and since \( \max |\chi(g)| = d_x \) we conclude from (4) and (5) that \( \chi_i(g) = d_i \) for all \( i \). It follows, that \( D_{\chi_i}(g) = E \) so \( \chi_i \in M^\perp \) and \( M^\perp \) is a subhypergroup which is normal. It is obvious that \( (M^\perp)^\perp \) is a closed normal subgroup containing \( M \). If \( N \) is some arbitrary closed normal subgroup containing \( M \), then \( (M^\perp)^\perp \subset N \) because if \( n \in (M^\perp)^\perp - N \) we can (by going to the factor group \( G/N \)) find a representation \( D \in \hat{G} \) such that \( D(g) = E \) for \( g \in N \) but \( D(n) \neq E \). Then \( D \in N^\perp - M^\perp \) which contradicts \( M \subseteq N \).

(ii) If \( \mathcal{S} \subseteq \hat{G} \) it is clear that \( \mathcal{S}^\perp \) is a closed normal subgroup and \( (\mathcal{S}^\perp)^\perp \) is a normal subhypergroup. The matrix elements from \( \mathcal{S} \) and \( I \) can be regarded as a family of continuous functions on \( G/\mathcal{S}^\perp \) which separates points. Let \( \mathfrak{R} \) be the set of linear combinations of matrix elements from the normal subhypergroup \( \mathcal{S}^* \) generated by \( \mathcal{S} \) and \( I \). By the Peter-Weyl theorem \( \mathfrak{R} \) is uniformly dense in the space of continuous functions on \( G/\mathcal{S}^\perp \). Now if there were a representation \( D \in \mathcal{S}^\perp \subseteq \mathcal{S}^* \), each matrix element \( a(g) \) from \( D \) could be uniformly approximated by elements of \( \mathfrak{R} \) but on the other hand \( a(g) \) is orthogonal to \( \mathfrak{R} \) by the orthogonality relations. This shows that \( \mathcal{S}^* = \mathcal{S}^\perp \).

(iii) Proof obvious.

3. Multipliers.

**Definition 3.1.** A hyperfunction on \( \hat{G} \) is a mapping which assigns to each \( \chi \in \hat{G} \) a linear transformation of a complex vector space of dimension \( d_\chi \).

**Definition 3.2.** A hyperfunction \( \Gamma \) on \( \hat{G} \) is called a multiplier if for each Fourier series for a continuous function

\[
f(g) \sim \sum d_\chi \text{ Tr } \langle A_\chi D_\chi(g) \rangle
\]
the series

\[ f_\Gamma(g) \sim \sum d_x \text{Tr} \langle \Gamma_x A_x D_x(g) \rangle \]

is also a Fourier series for a continuous function.

It is easy to see from the closed graph theorem that if \( \Gamma \) is a multiplier there exists a bounded measure \( \mu_\Gamma \) on \( G \) such that \( f_\Gamma = f \ast \mu_\Gamma \) (convolution product). Less trivial is the following extension of a theorem of Sidon:

**Theorem 2.** Let \( \Gamma \) be a multiplier such that for each Fourier series \((6)\) (with continuous \(f\)) the corresponding series \((7)\) is absolutely convergent. Then there exists a function \( F \in L^1(G) \) such that

\[ f_\Gamma = f \ast F \quad \text{for all continuous } f. \]

**Proof.** Let \( \Gamma \) be a multiplier with the properties stated in the theorem. Then \( B \Gamma \) is also of that type provided \( B \) is a hyperfunction on \( \hat{G} \) satisfying \( \sup_x \|B_x\| < \infty \). To see that \( B \Gamma \) really is a multiplier we remark that an absolutely convergent series (in the sense defined in this paper) is uniformly convergent on \( G \); this last fact is easily verified by writing each Fourier matrix \( A_x \) as \( P_x V_x \) where \( P_x \) is positive definite and \( V_x \) is unitary. Using a previous remark we see that there exist bounded measures \( \mu_\Gamma \) and \( \mu_{B\Gamma} \) on \( G \) such that

\[ f_{B\Gamma} = f \ast \mu_{B\Gamma}, \]

\[ f_\Gamma = f \ast \mu_\Gamma. \]

The hyperfunctions \( B \) satisfying \( \sup_x \|B_x\| < \infty \) (the bounded hyperfunctions) form a Banach space under the norm \( \sup_x \|B_x\| \). The mapping \( T: B \rightarrow \mu_{B\Gamma} \) is a linear mapping of the Banach space of bounded hyperfunctions into the Banach space of measures on \( G \) and again from the closed graph theorem it follows easily that this mapping is continuous. Now an integrable function on \( G \) can be identified with a hyperfunction on \( \hat{G} \) via the Fourier series expansion. If the function \( \phi \in L^1(G) \) corresponds to \( B \) in this manner we see that \( \mu_{B\Gamma} \) is absolutely continuous with respect to Haar measure and has a derivative, say \( \phi_\Gamma \in L^1(G) \). Then the mapping \( \tilde{T}: \phi \rightarrow \phi_\Gamma \) is a linear transformation of \( L^1(G) \) into itself and since \( T \) above is continuous it follows that \( \tilde{T} \) is spectrally continuous in the sense of [2]. Furthermore \( \phi_\Gamma = \mu_\Gamma \ast \phi \) so \( \tilde{T} \) commutes with right translations on \( G \).

This being established, Theorem 2 follows from Theorem A in [2] which is an extension of a theorem of Littlewood and states that the spectrally continuous operators that commute with right translations are precisely the left convolutions with \( L^2 \)-functions on \( G \).
Definition 3.3. A subset \( S \subseteq \hat{G} \) is called distinguished if for every Fourier series for a continuous function

\[
f(g) \sim \sum_{x \in G} d_x \text{Tr} \left( A_x D_x(g) \right)
\]

the subseries

\[
\sum_{x \in S} d_x \text{Tr} \left( A_x D_x(g) \right)
\]

also represents a continuous function \( f_S \).

For abelian groups \( G \), the distinguished sets were investigated in [1]. For the noncommutative case a partial description is given by

Theorem 3. The distinguished sets that preserve positivity in the sense that \( f_S \geq 0 \) whenever \( f \geq 0 \) are precisely the normal subhypergroups of \( \hat{G} \).

Proof. The mapping \( f \rightarrow f_S \) is continuous (uniform topology) by the closed graph theorem and commutes with left translations. Hence there exists a bounded measure \( \mu_S \) on \( G \) such that \( f_S = f * \mu_S \) for all \( f \). The mapping \( f \rightarrow f_S \) also commutes with right translations, so that the Fourier-Stieltjes series for \( \mu_S \) has the form

\[
\mu_S(g) \sim \sum_{x \in S} d_x \chi(g).
\]

Using the assumption of the theorem we see that \( \mu_S \) is a positive measure so by a theorem of Wendel [7] the condition \( \mu_S * \mu_S = \mu_S \) implies that there exists a compact subgroup \( K \) of \( G \) such that \( \mu_S(A) = \mu(A \cap K) \) for every Borel set \( A \), where \( \mu \) is the Haar measure on \( K \). From the Fourier-Stieltjes series for \( \mu_S \) we see that

\[
\int_G \overline{D}_x(g) d\mu_S(g) = \int_K \overline{D}_x(k) d\mu(k) = \begin{cases} E & \text{if } x \subseteq S, \\ 0 & \text{if } x \notin S \end{cases}
\]

which implies \( \overline{D}_x(k) = E \) for all \( k \) if and only if \( x \subseteq S \) or otherwise expressed: \( K^\perp = S \).

On the other hand, if \( S \) is a normal subhypergroup then the Haar measure \( \mu \) on \( S^\perp \) extended to a measure on \( G \) by \( \mu(E) = \mu(E \cap S^\perp) \) has the Fourier-Stieltjes series

\[
\mu(g) \sim \sum_{x \in S} d_x \chi(g)
\]

and for each continuous function \( f \) on \( G \) with Fourier series (8) the continuous function \( f * \mu \) has Fourier series (9), proving that \( S \) is distinguished.
4. Lacunary series. In this section we discuss extensions of theorems of Kolmogoroff and Banach.

Let $I$ be a set and to each element $i$ of $I$ attached an integer $d_i$. We consider the compact group $G = \prod_{i \in I} U(d_i)$ where $U(m)$ denotes the unitary group in $m$ dimensions. The projection $D_i$ of $G$ onto $U(d_i)$ is a unitary representation which clearly is irreducible, in other words $I$ can be regarded as a subset of $\hat{G}$. We consider Fourier series of the form

$$\sum_{i \in I} d_i \text{Tr} \langle A_i D_i(g) \rangle$$

and we shall now indicate the proof of the following theorem.

**Theorem 4.** Suppose $f \in L^1(G)$ and has a Fourier series of the form (10). Then $f \in L^2(G)$ and moreover $2^{-1/2} \| f \|_2 \leq \| f \|_1 \leq \| f \|_2$.

In the case where $d_i = 1$ for each $i \in I$, this result is a well known theorem of Kolmogoroff.

The essence of Theorem 4 is proved in [2]. In fact let us consider a finite subset $J$ of $I$ and a series of the form

$$s(g) = \sum_{j \in J} d_j \text{Tr} \langle B_j D_j(g) \rangle.$$ 

It is then clear that

$$\int_G |s(g)|^m \, dg = \int_{V_J} \left| \sum_{j} d_j \text{Tr} \langle B_j D_j \rangle \right|^m \, dD_J$$

where $dD_J$ denotes the Haar measure on the product $V_J = \prod_{j \in J} U(d_j)$. By the proof of Lemma 4.1 in [2] and the relation (4.12) in [2] we get

$$\int_G |s(g)|^4 \, dg \leq 2 \left[ \int_G |s(g)|^2 \, dg \right]^2.$$ 

Using the inequality

$$\left[ \int |h(g)| \, dg \right]^2 \leq \left[ \int |h(g)|^4 \, dg \right]^{-1} \left[ \int |h(g)|^2 \, dg \right]$$

which is a special case of Hölder's inequality we obtain

$$\int_G |s(g)| \, dg \geq 2^{-1/2} \left[ \int_G |s(g)|^2 \, dg \right]^{1/2}.$$ 

By standard approximation arguments the function $f$ can be ap-
proximated in $L^1$-norm by functions of the form (11), and (13) becomes valid for the function $f$. Theorem 4 is proved.

We shall now see that the set $I$ in the above situation appears as a lacunary subset of $\hat{G}$ in a certain sense.

Let again $G$ be an arbitrary compact group. If $\alpha, \beta \in \hat{G}$ we denote by $D_{\alpha}$ and $D_{\beta}$ arbitrary members of $\alpha$ and $\beta$ respectively, $d_{\alpha}$ and $d_{\beta}$ the corresponding dimensions and $n_{\alpha,\beta}$ the number of irreducible components in $D_{\alpha} \otimes D_{\beta}$ (counted with multiplicity).

**Definition 4.1.** A subset $S \subset \hat{G}$ is called lacunary if the two following conditions are satisfied.

(I) Whenever $(\alpha, \beta)$ and $(\gamma, \delta)$ are different pairs from $S$ (that is, the characters $\alpha + \beta$ and $\gamma + \delta$ are different) $D_{\alpha} \otimes D_{\beta}$ and $D_{\gamma} \otimes D_{\delta}$ are disjoint.

(II) There exists a constant $K$ such that $n_{\alpha,\beta} < K$ for all $\alpha, \beta \in S$.

A Fourier series of the form $\sum_{x \in S} d_x \text{Tr} (A_x D_x(g))$ is called lacunary if $S$ is lacunary.

The following statements show that the series (10) is indeed lacunary.

(i) $D_i \otimes D_j$ is irreducible if $i \neq j$.

(ii) If $d_i = 1$ then $D_i \otimes D_i$ is irreducible.

(iii) If $d_i \geq 2$ then $D_i \otimes D_i$ decomposes into two irreducible parts (of dimensions $(d_i^2 + d_i)/2$ and $(d_i^2 - d_i)/2$).

(iv) $D_i \otimes D_i$ is disjoint from $D_j \otimes D_j$ if $i \neq j$.

(i) and (ii) are obvious. (iii) is a corollary of Lemma 4.1 in [2] combined with the fact that the space of symmetric and antisymmetric tensors are left invariant by $D_i \otimes D_i$. Concerning (iv) we remark that the number of irreducible components common to $D_i \otimes D_i$ and $D_j \otimes D_j$ is equal to

$$\int_{G}(x_i \bar{x_i})^2 dg = \int_{U(d_m) \times U(d_i)} (\text{Tr} D_i)^2 (\text{Tr} D_j^{-1})^2 dD_i dD_j.$$ 

($dD_m$ is the Haar measure on $U(d_m)$), and this last integral vanishes as shown in the proof of the cited lemma.

For a general compact group we have a simple result in similar direction.

**Theorem 5.** Let $f$ be a central function in $L^1(G)$ and suppose $f$ has a lacunary Fourier series. Then $f \in L^2(G)$.

**Proof.** The Fourier expansion of $f$ can be written $f(g) \sim \sum_{x \in S} a_x \chi_x(g)$ where $a_x$ is a complex number and $S$ is lacunary. We consider a finite partial sum $s(g) = \sum_{i=1}^{N} a_x \chi_x(g)$. Then
\[
S^2 = \sum a_p \chi_p^2 + 2 \sum_{p<q} a_p a_q \chi_p \chi_q.
\]

If we here expand \( \chi_p^2 \) and \( \chi_p \chi_q \) into a sum of characters the same \( \chi_i \) will not occur more than once due to condition (I), and we get easily

\[
\int_G |s(g)|^4 dg = \sum_{i=1}^N |a_p|^4 n_{p,p} + 2 \sum_{p>q} |a_p|^2 |a_q|^2 n_{p,q} \leq K \left( \sum_{i=1}^N |a_p|^2 \right)^2.
\]

Using the inequality (12) we obtain the conclusion \( f \in L^2(G) \) in exactly the same manner as before.

Concerning the relation with the classical definition of lacunary series we remark that a series of the form \( \sum a_k e^{i n_k x} \) where \( n_{k+1}/n_k > 2 \) for all \( k \), is lacunary in the sense of Definition 4.1. This is easily verified and is indeed a basic property in the proof of most classical theorems on lacunary series.

As a simple consequence of Theorem 5 we mention the following fact:

**Theorem 6.** Let \( G \) be an abelian group which is compact and not totally disconnected. Suppose the infinite series

(14) \[
\sum a_{x \sigma} \chi(g)
\]

is a Fourier series for some \( L^1 \)-function for each permutation \( \sigma \) of \( \hat{G} \). Then \( \sum_{x \in \hat{G}} |a_x|^2 < \infty \).

**Proof.** By a theorem of Pontrjagin \( \hat{G} \) has an infinite cyclic subgroup and therefore an infinite countable lacunary subset. Now we can write

\[
\sum_{a_{x \neq 0}} a_{x \chi}(g) = \sum_S a_{x \chi}(g) + \sum_T a_{x \chi}(g)
\]

where both sets \( S \) and \( T \) are infinite and \( \sum_S |a_x| < \infty \). Hence \( \sum_T a_{x \sigma} \cdot \chi(g) \) is an \( L^1 \)-series for every permutation \( \sigma \) of \( \hat{G} \). Choose this permutation in such a way that \( \sum_T a_{x \sigma} \cdot \chi(g) \) is a lacunary series and apply Theorem 5. Q.E.D.

If \( G \) is an arbitrary compact group, \( \hat{G} \) need not possess any infinite lacunary subsets. As a simple example we mention \( G = SU(2) \). The hypergroup structure of \( \hat{G} \) is here described by the Clebsch-Gordan formula [8] which shows that the requirement (II) in Definition 4.1 is not fulfilled for any infinite subset \( S \subset \hat{G} \).
Bibliography


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