A CHARACTERIZATION OF LIGHT OPEN MAPS OF EUCLIDEAN SPACES INTO EUCLIDEAN SPACES

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We will use the term open map to mean a map $f$ of $X$ into $Y$ such that $f(U)$ is open in $Y$ for every open set $U$ in $X$. A map $f$ of $\mathbb{R}^n$ (Euclidean $n$-space) into $\mathbb{R}^m$ is pseudo-monotone if and only if $\mathbb{R}^n - f^{-1}(X)$ has no bounded component for every closed set $X$ in $\mathbb{R}^n$ such that $\mathbb{R}^m - X$ has no bounded component.

The purpose of this note is to prove that a light map $f$ of $\mathbb{R}^n$ into $\mathbb{R}^m$ is open if and only if $f$ is pseudo-monotone. But first we must prove the following lemma:

**Lemma.** If $f$ is an open map of $\mathbb{R}^n$ into $\mathbb{R}^m$, $U$ is a bounded open set in $\mathbb{R}^n$, and $f(U) \cap W \neq \emptyset$ for some component $W$ of $\mathbb{R}^m - f(\text{bdry } U)$, then $W \subseteq f(U)$. Thus, in particular, $f(U)$ does not intersect the unbounded component of $\mathbb{R}^m - f(\text{bdry } U)$.

**Proof.** Assume $W \subseteq f(U)$. Let $p$ be a point of $W - f(U)$ and $q$ be a point of $W \cap f(U)$. Let $pq$ be an arc from $p$ to $q$ in $\mathbb{R}^n - f(\text{bdry } U)$. Since $q$ is in the compact set $f(U)$ and $p$ is not, there is a first point $x$ on the arc $pq$ in the order $p$ to $q$ such that $x$ is in $f(U)$. We note $x$ is not in $f(\text{bdry } U)$ since $pq \cap f(\text{bdry } U) = \emptyset$. Therefore $f^{-1}(x) \cap U \neq \emptyset$ and $x \subseteq \text{bdry } f(U)$. Hence $f$ is not open.

**Remark.** If $W$ is the union of $X$ and the bounded components of $\mathbb{R}^m - X$, then $\mathbb{R}^m - W$ has no bounded components.

**Theorem.** A light map $f$ of $\mathbb{R}^n$ into $\mathbb{R}^m$ is open if and only if $f$ is pseudo-monotone.

**Proof.** Assume $f$ is open but not pseudo-monotone. Then by definition of pseudo-monotone there is a closed set $X$ in $\mathbb{R}^m$ such that $\mathbb{R}^m - X$ has no bounded component and $\mathbb{R}^n - f^{-1}(X)$ has a bounded component. Let $\mathbb{R}^n - f^{-1}(X) = A \cup B$ where $A$ and $B$ are open, $A \cap B = \emptyset$, and $A$ is both bounded and nonempty. For the open set $A$ we have $f(\text{bdry } A) \subseteq X$ and $f(A) \subseteq \mathbb{R}^m - X$. Hence $f(A) \cap W \neq \emptyset$ for some unbounded component $W$ of $\mathbb{R}^m - X$. ($\mathbb{R}^m - X$ has no bounded component.) But $\mathbb{R}^m - X \subseteq \mathbb{R}^m - f(\text{bdry } A)$. Therefore $f(A) \cap W' \neq \emptyset$,

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where \( W' \) is the unbounded component of \( \mathbb{R}^m - f(\text{bdry } A) \) containing \( W \). By our lemma this is not possible.

Assume \( f \) is pseudo-monotone but not open. Then there is an open set \( U \) in \( \mathbb{R}^n \) such that \( f(U) \) is not open in \( \mathbb{R}^m \). Let \( p \) be a point of \( f(U) \cap \text{bdry } f(U) \), let \( q \) be a point of \( f^{-1}(p) \cap U \), and let \( V \) be a bounded neighborhood of \( q \) lying in \( U \) such that the boundary of \( V \) does not intersect the 0-dimensional set \( f^{-1}(p) \). The boundary of \( V = B \) is compact, hence there is a spherical neighborhood \( S \) of \( p \) which does not intersect the compact set \( f(B) \). Let \( x \) be a point of the set \( S - f(U) \). \( S - f(U) \) is not empty because \( p \) is the boundary of \( f(U) \). Let \( b \) be the first point on the line segment from \( x \) to \( p \) which is in the compact set \( f(V) \). Then \( b \) is in \( f(V) \) since \( b \) is in \( S \) and \( S \cap f(\text{bdry } V) = \emptyset \). Let \( a \) be a point of \( f^{-1}(b) \cap V \). Let \( r \) denote the ray from \( p \) through \( x \) and let \( r' \) denote the subray of \( r \) with source at \( x \). Let \( S' \) denote the open set \( S - r' \). Let \( V' \) be a neighborhood of \( a \), whose closure lies in the open set \( f^{-1}(S) \cap V \), and whose boundary does not intersect the 0-dimensional set \( f^{-1}(b) \). \( b \) is in \( f(V') \cap r \) and is the only point of the ray \( r' \cup s \) which lies in \( f(V') \). Hence \( b \) is in an unbounded component of \( \mathbb{R}^m - f(\text{bdry } V') \). Let \( W \) be the union of \( f(\text{bdry } V') \) and the bounded components of \( f(\text{bdry } V') \), then \( \mathbb{R}^m - W \) has no bounded components. Consider the set \( f^{-1}(W) \). \( a \in f^{-1}(W) \) because \( b \in W \). But \( \text{bdry } V' \subset f^{-1}(W) \); hence the bounded set \( V' \subset f^{-1}(W) \) and therefore \( a \in f^{-1}(W) \).

In showing that an open map is pseudo-monotone we did not use the fact that \( f \) was light. Hence an open map of \( \mathbb{R}^n \) into \( \mathbb{R}^m \) is pseudo-monotone.

In comparing the theorem with a result of A. D. Wallace [2], the referee points out that the following proposition is false:

**Proposition.** If \( f \) is a pseudo-monotone map of \( \mathbb{R}^n \) into \( \mathbb{R}^m \), then there exists an increasing sequence \( C_1, C_2, \cdots \) of \( n \)-cells such that \( UC_i = \mathbb{R}^n \) and \( f/C_i \) is quasi-monotone.

**Bibliography**


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