RINGS OF ZERO-DIVISORS

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1. Introduction. A well known theorem of algebra states that any integral domain can be embedded in a field. More generally [2, p. 39 ff.], any commutative ring $R$ with unit-element can be embedded in a ring $S$ with the same unit-element as $R$, such that any element of $S$ is either invertible or the product of an invertible element by a zero-divisor of $R$. The following statement can, in a very rough sense, be regarded as a "dual":

Any commutative ring $R$ with unit-element can be embedded in a ring $S$ with the same unit-element as $R$ such that every element of $S$ is either a zero-divisor or an invertible element of $R$.

This statement will be proved in §2 (Theorem 1) and as in the theorem quoted we have the corollary that any commutative ring with a unit-element 1, which has no invertible elements other than 1, can be embedded in a ring in which every element $\neq 1$ is a zero-divisor. We are thus led to consider commutative rings with unit-element 1, in which every element $\neq 1$ is a zero-divisor. Such rings will be called $O$-rings for short. It is clear that every Boolean ring with a unit-element is an $O$-ring, and Theorem 1 shows without difficulty that $O$-rings exist which are not Boolean; this answers a question raised by Kaplansky.\(^1\) An $O$-ring may be regarded as an algebra over the field of two elements and thus is a special case of an $O$-algebra, viz. an algebra with unit-element over a field $F$ in which every element not in $F$ is a zero-divisor. Again any algebra with no invertible elements other than those of $F$ can be embedded in an $O$-algebra, and some theorems on the structure of these algebras are proved in §3. In particular, any $O$-algebra $R$ over a field $F$ is a subdirect product of extension fields of $F$; the number of components is infinite unless $R = F$ or $R$ is Boolean. Moreover, in any representation of $R$ as such a subdirect product, and for any element $a$ of $R$ and any equation (in one indeterminate) over $F$, there are infinitely many components of $a$ satisfying this equation, unless $a$ is a scalar or $F$ is the field of two elements and $a$ is idempotent. Nevertheless, (non-Boolean) $O$-rings exist which are countable. It is also noted that any $O$-algebra, which is regular in the sense of von Neumann, is of dimension 1 or Boolean.

\(^1\) Communicated to the writer by M. P. Drazin. It was this question which gave rise to the present note.

Received by the editors March 14, 1958.

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2. **The embedding theorem.** Throughout this note, we take "ring" to mean "commutative ring with unit-element"; further, to say "$P$ is a subring of $R$" is understood to mean "subring with the same unit-element," and similarly, to embed $R$ in a ring $S$ means to embed as a subring, i.e. with the same unit-element. The common unit-element will always be denoted by $1$. These conventions apply in particular when $R$ is an algebra over a field $F$, so that in this case $F$ may be considered as a subalgebra of $R$.

The result to be proved may now be stated as follows:

**Theorem I.** Any ring $R$ may be embedded in a ring $S$ such that any element $s$ of $S$ is a zero-divisor unless $s \in R$ and $s$ has an inverse in $R$. If $R$ is an algebra over a field $F$ then $S$ may also be taken to be an algebra over $F$.

The essential step towards establishing this theorem is the proof of

**Lemma 1.** Let $R$ be any ring and $a$ any element of $R$ which has no inverse in $R$. Then there exists a ring $R'$ containing $R$ as a subring such that

(i) $R'$ contains an element $a' \neq 0$ satisfying $aa' = 0$,

(ii) any invertible element of $R'$ belongs to $R$ and is already invertible in $R$.

**Proof.** Let $R[x]$ be the ring of polynomials in an indeterminate $x$ with coefficients in $R$. Since $R$ has a 1, every proper ideal of $R$ is contained in a maximal ideal. Now $a$ has no inverse, therefore the principal ideal generated by $a$ is proper and hence is contained in a maximal ideal, $M$ say, of $R$. Let $M'$ be the ideal of $R[x]$ generated by the elements $mx(m \in M)$. Then $M'$ consists precisely of the elements $mxu (m \in M, u \in R[x])$, and it follows that

\[(1) \quad M' \cap R = 0,\]

\[(2) \quad x \in M'.\]

Equation (1) is clear, since an equation of the form

\[b = mxu, \quad b \in R, m \in M, u \in R[x]\]

implies $b = 0$, by a comparison of the terms of degree zero. To prove (2), let us suppose that $x \in M'$, so that

\[(3) \quad x = mxu,\]

where $m \in M$ and $u = u_0 + u_1 x + \cdots + u_n x^n (u_i \in R)$, say; by com-
paring the terms of degree 1 in (3) we obtain $1 = \mu u_0$, which contradicts the fact that $M$ is proper.

We now put $R' = R[x]/M'$. By (1), the natural homomorphism of $R[x]$ onto $R'$, when restricted to $R$, is one-one; we may therefore identify $R$ with a subring of $R'$. If we write $a'$ for the image of $x$ under the natural homomorphism, then (2) shows that $a' \neq 0$, while $aa' = 0$, because $ax \in M$. It only remains to show that any invertible element of $R'$ belongs to $R$.

Let $u'$ then be an element of $R'$ with inverse $v'$, so that $u'v' = 1$. Going back to $R[x]$, we find elements $u, v$ mapping onto $u', v'$ respectively, i.e. we have an equation of the form

\[ uv = 1 + xmw \quad w \in R[x]. \tag{4} \]

Let $u = \sum u_i x^i, v = \sum v_i x^i, w = \sum w_i x^i$, where $u_i, v_i, w_i \in R$. If $u$ is of degree 0, then $u \in R$, and by comparing the terms independent of $x$ in (4) we find that $w_0 = 1$; thus $u$ belongs to $R$ and is invertible in $R$, and hence the same is true for $u'$. In what follows we may therefore suppose that $u$ is of positive degree, $r$ say. By symmetry, $v$ may also be taken to have positive degree, $s$ say. Now $ux'^t \in M'$ whenever $ur \in M$, and since we are only interested in the residue class of $u \pmod{M'}$, we may assume $u_r \in M$. Similarly $v_s \in M$ and since $M$ is maximal, $u_r, v_s \in M$. Equating coefficients of $x^{r+s}$ in (4) we find

\[ u_r v_s = mw_{r+s-1} \in M, \]

a contradiction. This shows that all the invertible elements of $R'$ belong to $R$ and have their inverses in $R$, and the lemma is established.

It is now easy to prove Theorem 1, using a Steinitz tower construction. Given any ring $R$ and any $a \in R$ which has no inverse we can by Lemma 1, embed $R$ in a ring in which $a$ becomes a zero-divisor, without increasing the set of invertible elements. By transfinite induction we can embed $R$ in a ring $R'$ without increasing the set of invertible elements, and such that every element of $R$ which is not invertible becomes a zero-divisor in $R'$. Starting with $R_0 = R$, we now define $R_n$ inductively by the equation $R_{n+1} = R_n'$. In this way we obtain an ascending sequence of rings

\[ R \subset R_1 \subset R_2 \subset \cdots. \]

The union $S$ of these rings is again a ring; this ring contains $R$ and by construction, every invertible element of $S$ belongs to $R$ and is invertible in $R$. Further, any element of $S$ which is not invertible belongs to $R_n$ for some $n$, and hence is a zero-divisor in $R_{n+1}$; a fortiori it is a zero-divisor in $S$, and this completes the proof of the theorem.
in the case of rings. If $R$ is an algebra over a field $F$, all the rings occurring become algebras over $F$ and the proof goes through unchanged.

From the proof of Theorem 1 it is clear that $S$ has the same cardinal as $R$ or $\aleph_0$, whichever is the greater. In particular, if $R$ is countable, then so is $S$.

If in Theorem 1 we replace elements by finite subsets, we obtain the following generalization:

Theorem 1'. Any ring $R$ may be embedded in a ring $S$ such that
(i) $S$ has no invertible elements other than those of $R$, (ii) every proper ideal of $R$ generates a proper ideal of $S$ and (iii) every finitely generated proper ideal of $S$ has a nontrivial annihilator.

Clearly this includes Theorem 1, since an element $a$ of $S$ either generates a proper ideal, in which case it is a zero-divisor, by (iii), or it is invertible and then both $a$ and its inverse belong to $R$, by (i).

As we shall not have occasion to use Theorem 1', we omit the proof which is very similar to that of Theorem 1.

3. Algebras without nontrivial units and $O$-algebras. Let $R$ be a ring in which no element $\neq 1$ is invertible. Since $-1$ necessarily has an inverse, we have $-1 = 1$, whence $2x = 0$ for all $x \in R$. Therefore $R$ can be regarded as an algebra over $F = GF(2)$, the field of two elements, and $R$ has no "units" other than the nonzero elements of $F$. More generally, we shall say that an algebra $R$ over a field $F$ has no nontrivial units if the only invertible elements of $R$ are the nonzero elements of $F$. If moreover, all the elements of $R$ except those in $F$ are zero-divisors, we call $R$ an $O$-algebra. From Theorem 1 and the remark following it we now obtain

Theorem 2. Any algebra $R$ over $F$ without nontrivial units may be embedded in an $O$-algebra $S$. If $R$ is countable, so is $S$.

Taking the groundfield to be $GF(2)$, we obtain the

Corollary. Any ring with no invertible elements other than 1 may be embedded in an $O$-ring, which is countable if the given ring was countable.

Returning to algebras without nontrivial units, we have the following structure theorem.

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* We recall that $R$ contains 1 and hence all the elements of $F$. The observation that the arguments as originally presented for the case $F = GF(2)$ carry over to the general case is due to the referee.
Theorem 3. Let $R$ be an algebra over $F$ without nontrivial units. Then $R$ is a subdirect product\(^4\) of extension fields of $F$, and every element $x$ of $R$ which is not in $F$ is transcendental over $F$, unless $F = GF(2)$ and $x$ is idempotent.\(^5\) If, moreover, $R$ has finite dimension over $F$, then either $R = F$ or $R$ is a Boolean algebra.

Proof. We begin by showing that $J$, the Jacobson radical of $R$, is zero. Let $a \in J$, then $1 + a$ has an inverse and so belongs to $F$, whence $a \in F$; this shows that $J \subseteq F$. But $1 \not\in J$, so that $J$ is a proper ideal in $R$ and therefore also in $F$, whence $J = 0$. It follows\(^6\) that $R$ is a subdirect product of fields which in our case are algebras over $F$, i.e. extension fields of $F$. If $R$ is of finite dimension then the number of factors is finite and $R$ is in this case a direct product of extension fields of $F$.\(^7\) Since $R$ has no nontrivial units, this is possible only if (i) $R = F$ or (ii) $F = GF(2)$ and $R$ is the direct sum of a finite number of copies of $F$, i.e. if $R$ is a Boolean algebra. This proves the last part; to complete the proof, we consider an element $x$ of $R$ which is algebraic over $F$. Then the subalgebra $F[x]$ generated by $x$ is finite-dimensional and has no nontrivial units, whence by the part just proved, either $F[x] = F$, i.e. $x \in F$, or $F[x]$ is Boolean, in which case $F = GF(2)$ and $x^2 = x$, which is what we wished to show.

It is clear that a Boolean algebra is a $O$-ring, since each $x$ satisfies $x(1 - x) = 0$. To obtain an $O$-ring which is not Boolean, we take a non-Boolean algebra over $GF(2)$ without nontrivial units, e.g. the algebra of polynomials in a single variable over $GF(2)$. By Theorem 2, Corollary, this algebra may be embedded in an $O$-ring, which is not Boolean since it contains elements which are not idempotent. Moreover it is countable. In a similar way we obtain for a general field $F$ an $O$-algebra different from $F$ and of cardinal max$(\aleph_0, \text{card } F)$.

We note that an algebra $R$ over $F$ without nontrivial units which is regular\(^8\) either coincides with $F$ or is Boolean. For by the regularity there exists for each $a \in R$ an element $x \in R$ such that $xa^2 = a$. Hence $(1 + a(1 - x))(1 - xa(1 - x)) = 1$, so that $1 + a(1 - x) = \gamma \in F$. Now $a = \gamma + ax - 1$, and multiplying by $a$ we find that $a^2 = \gamma a + xa^2 - a = \gamma a$, which shows that $a$ is algebraic over $F$; the result stated now follows from Theorem 3.

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\(^4\) The term "product" is used here in the sense of Bourbaki.

\(^5\) I am indebted to the referee for the assertion about elements of $R$, which allows some of the later proofs to be shortened.

\(^6\) McCoy [1, p. 135].

\(^7\) This follows from Theorem 32 of McCoy [1, p. 125].

\(^8\) I.e., to every $a \in R$ there corresponds an $x \in R$ satisfying $axa = a$. M. P. Drazin has proved, somewhat more generally, that in any (not necessarily commutative) ring with 1 and no other invertible elements, every regular element is idempotent.
From the proof of the last assertion of Theorem 3 we see that for an algebra over $F$ without nontrivial units which is not equal to $F$ itself or Boolean, any representation as a subdirect product must contain infinitely many factors. In the case of $O$-algebras we can say rather more.

**Theorem 4.** Let $R$ be any $O$-algebra over $F$, and $a$ an element of $R$ which is transcendental over $F$. Then in any representation of $R$ as a subdirect product of fields and for any equation

$$f(\xi) = 0$$

of positive degree over $F$, there are infinitely many components of $a$ satisfying (5).

An $O$-algebra has no nontrivial units, so that $R$ can be written as a subdirect product of fields, by Theorem 3. This shows that the assertion is never vacuous. To prove it, we consider first the special case $f(\xi) = \xi$, i.e. we show that infinitely many components of $a$ are zero. For if there is a transcendental element of $R$ with only finitely many components equal to zero, let $a = (a_i)$ $(i \in I)$ be such an element, with components $a_i$, where $a_1 = a_2 = \cdots = a_k = 0$, $a_i \neq 0$ $(i \neq 1, 2, \ldots, k)$. We may suppose $a$ chosen so that $k$ has its least value ($k \geq 0$). Since $a \in F$, there exists $b \in R$, $b \neq 0$, such that $ab = 0$. This equation shows that for each $i \in I$ either $a_i = 0$ or $b_i = 0$. In particular $b_i = 0$ except possibly for $i = 1, \ldots, k$, and since $b \neq 0$, some $b_i$ must be different from zero, which shows that $k > 0$. Now the element $a + b$ has fewer than $k$ components equal to zero, since $a_i + b_i = 0$ only if $a_i = b_i = 0$. By the hypothesis on $k$, $a + b$ is algebraic over $F$; therefore either $a + b = \gamma \in F$ and $ab = a^2 - \gamma a = 0$, or $F = GF(2)$ and $a + b$ is idempotent. But then each $a_i$ (and each $b_i$) is either 0 or 1, so that we have $a^2 = a$. In either case we have found an equation satisfied by $a$, which contradicts the fact that $a$ was chosen transcendental.

To complete the proof, suppose that $a$ has only finitely many components satisfying (5). This means that $f(a)$ has only finitely many components equal to zero, and by what has been proved, $f(a)$ is algebraic over $F$, i.e. either $f(a) = \gamma \in F$ or $f(a)^2 = f(a)$. In either case $a$ is algebraic over $F$, and this establishes the theorem.

**References**


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