THE SIMPLICITY OF CERTAIN NONASSOCIATIVE ALGEBRAS

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Introduction. Let $V$ be an additively written elementary $p$-group of order $p^n$, where $n>1$, and let $\phi(u, v)$, $\psi(u, v)$ be two alternate biadditive functions defined on $V$ and with values in a field $K$ of characteristic $p$. Let $k$ be a fixed nonzero element in $V$, and assume that the following two conditions are satisfied:

(0.1) If $\alpha, \beta$ are nonzero elements in $K$ then $\alpha \phi(u, v) + \beta \psi(u, v) = 0$ for all $v$ implies $u = 0$;

(0.2) $\phi(k, v) = 0$ for all $v$.

If $\phi$ is identically zero, then the conditions (0.1)–(0.2) are equivalent to saying that $\psi$ is nondegenerate. Define an algebra $L$ over $K$ with basis

$$\{e_u \mid u \in V, u \neq 0, u \neq k\}$$

by the multiplication table:

$$e_u e_v = \phi(u, v)e_{u+v} + \psi(u, v)e_{u+v+k},$$

where we set $e_0 = e_k = 0$. The purpose of this note is to show that the algebra $L$ is simple if $p > 2$.

An elementary computation shows that

$$(e_u e_v)e_w = (e_u e_w)e_v + (e_v e_u)e_w$$

$$= (\psi(u, v)\psi(k, w) + \psi(v, w)\psi(k, u) + \psi(w, u)\psi(k, v))e_{u+v+w+2k}.$$ 

From this we may deduce easily that $L$ is a Lie algebra if and only if there exist two additive functions $f$ and $g$ on $V$ with values in $K$ such that

(0.3) $\psi(u, v) = f(u)g(v) - f(v)g(u)$

for all $u, v$ in $V$. In the case where $\psi(u, v)$ is given as in (3), therefore, the algebra $L$ is a special case of the algebras considered by Richard Block [2] if the following conditions are satisfied: $V$ is the direct sum of subgroups $V_1$ and $V_2$ such that $k \in V_1$; $\phi(u, v) = 0$ for all $u \in V_2$, $v \in V$; $\psi(u, v) = 0$ for all $u \in V_1$, $v \in V$. If $\psi(u, v)$ cannot be expressed as in (3), then $L$ is of course not a Lie algebra. But it gives an interesting family of simple nonassociative algebras.

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1. **The center of** \( L \). As a first step in proving the simplicity of \( L \) we show that the center of \( L \) is zero. Since \( V \) is an elementary \( p \)-group, we may express \( V \) as the direct sum of a subgroup \( U \) and the subgroup generated by \( k \). Then any element \( x \) in \( L \) can be written in the form

\[
(1.1) \quad x = \sum_{u \in U} \sum_{i=0}^{p-1} \alpha_{u,i} e_{u+ik}
\]

with \( \alpha_{u,i} \) in \( K \), where \( \alpha_{00} = \alpha_{01} = 0 \). For the element \( x \) given by (1.1) define \( x_u \) for each \( u \) in \( U \) by

\[
x_u = \sum_{i=0}^{p-1} \alpha_{u,i} e_{u+ik}.
\]

Then \( x = \sum x_u \), and if \( x \) is in the center of \( L \) then for each \( u \) in \( U \) the element \( x_u \) is also in the center of \( L \). Therefore if the center of \( L \) is not zero, then there must be in the center a nonzero element \( x \) of the form

\[
(1.2) \quad x = \alpha_0 e_u + \alpha_1 e_{u+k} + \cdots + \alpha_{p-1} e_{(p-1)k},
\]

where \( \alpha_i \in K \). For the element \( x \) given by (1.2) define the length \( \lambda(x) \) of \( x \) to be the number of nonzero coefficients \( \alpha_i \). Let the above \( x \) be of minimal length among the nonzero central elements such that \( x = x_u \) for some \( u \). We consider the following two cases:

**Case I.** \( u = 0 \). Then \( x \) is of the form

\[
x = \alpha_2 e_{2k} + \alpha_3 e_{3k} + \cdots + \alpha_{p-1} e_{(p-1)k}.
\]

Take an element \( v \in U \) such that \( \psi(k, v) \neq 0 \). Then

\[
x e_v = \alpha_2 \psi(2k, v) e_{3k} + \cdots + \alpha_{p-1} \psi((p-1)k, v) e_v = 0.
\]

Hence \( \alpha_2 = \cdots = \alpha_{p-1} = 0; x = 0 \), a contradiction.

**Case II.** \( u \neq 0 \). We have

\[
x e_{2k} = \sum_{i=0}^{p-1} \alpha_i \psi(u + ik, 2k) e_{u+(i+2)k} = 0.
\]

Since \( u + (i+2)k \neq 0 \), \( k \) for all \( i \), and since some \( \alpha_i \neq 0 \), we have \( \psi(k, u) = 0 \). Let \( v \in V \) be such that \( \psi(k, v) \neq 0 \). Then for every \( i \), \( u+v+ik \) is neither zero nor \( k \), since \( \psi(k, u+v+ik) \neq 0 \). Now we have

\[
x e_v = \sum_{i=0}^{p-1} (\alpha_i \phi(u, v) + \alpha_{i-1} \psi(u + (i - 1)k, v)) e_{u+v+ik},
\]

where \( \alpha_{-1} = \alpha_{p-1} \). Since \( x e_v = 0 \), we have
\[ (1.3) \quad \alpha_i \phi(u, v) + \alpha_{i-1} \psi(u + (i - 1)k, v) = 0 \]

for all \( i \). If \( w \) is an element such that \( \psi(k, w) = 0 \) then \( \psi(k, v+w) \neq 0 \). Hence we may replace \( v \) in (1.3) by \( v+w \). Then it follows that (1.3) is true for all \( v \in V \) and all \( i \). Let \( i = 1 \) in (1.3) and use (0.1). Then we have \( \alpha_0 = 0 \) or \( \alpha_1 = 0 \). Take an \( i \) such that \( \alpha_i \neq 0, \alpha_{i-1} = 0 \). Then from (1.3) it follows that

\[ (1.4) \quad \phi(u, v) = 0 \]

for all \( v \) in \( V \). Let \( j \) be such that \( \alpha_j = 0, \alpha_{j-1} \neq 0 \). Then from (1.3) we have

\[ (1.5) \quad \psi(u + (j - 1)k, v) = 0 \]

for all \( v \) in \( V \). Also from (1.4) we have \( \phi(u + (j - 1)k, v) = 0 \) for all \( v \in V \). Hence we have \( u + (j - 1)k = 0 \), a contradiction.

2. Minimal elements. For \( x \) in \( L \) denote by \( \rho(x) \) the number of nonzero \( x_w \), where \( u \in U \). Let \( M \) be a fixed nonzero ideal of \( L \), and call a nonzero element \( x \) in \( M \) minimal if \( \rho(x) \) is minimal. First we prove

**Lemma 2.1.** There exists a minimal element \( x \) in \( M \) for which \( x_0 \neq 0 \).

**Proof.** We shall derive a contradiction by assuming that the lemma is not true. Let \( x = \sum x_u \) be a minimal element. If there exists an element \( u \) in \( U \) such that \( x_u \neq 0 \) and \( \psi(k, u) \neq 0 \), then for any integer \( j \) the element \( u+jk \) is neither 0 nor \(-k\). Hence we may consider the product \( y = e_{u+jk}x \). It is readily seen that \( \psi(k, u) \neq 0 \) implies that \( y_0 \neq 0 \) for some \( j \). Clearly \( \rho(y) \leq \rho(x) \) and \( y \in M \). Hence it follows from our assumption that, for any minimal element \( x \), \( x_u \neq 0 \) implies \( \psi(k, u) = 0 \). Take an element \( v \) in \( V \) such that \( \psi(k, v) \neq 0 \). Then \( x_u e_v = 0 \) for any minimal element \( x \), because for \( y = x e_v \) we have \( \rho(y) \leq \rho(x) \) and if \( y_w \neq 0 \) then \( w = u+v \), where \( \psi(k, u) = 0 \), and hence \( \psi(k, w) \neq 0 \) which contradicts the conclusion obtained above. Now consider elements \( v' \) in \( V \) such that \( \psi(k, v') = 0 \), \( v' \neq 0 \), \( v' \neq k \). Then \( \psi(k, v+v') \neq 0 \), and hence \( x_u e_{v+v'} = 0 \). Let

\[ (2.1) \quad x_u = \alpha_0 e_u + \alpha_1 e_{u+k} + \cdots + \alpha_{p-1} e_{u+(p-1)k}. \]

Then \( x_u e_v = 0 \) implies

\[ \alpha_i \phi(u, v) + \alpha_{i-1} \psi(u + (i - 1)k, v) = 0 \]

for all \( i \), where we set \( \alpha_{-1} = \alpha_{p-1} \), because \( u+v+ik \) is neither zero nor \( k \). Similarly we have

\[ \alpha_i \phi(u, v+v') + \alpha_{i-1} \psi(u + (i - 1)k, v+v') = 0 \]

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and hence from (2.1) we have
\[ \alpha_i\phi(u, v') + \alpha_{i-1}\psi(u + (i - 1)k, v') = 0 \]
for all \( i \), which shows that \( x_u e_{v'} = 0 \). Thus we have shown that \( x \) belongs to the center of \( L \). By the result in the first section, \( x = 0 \). This is a contradiction. Thus Lemma 2.1 is proved.

Let \( x \) be a minimal element in the ideal \( M \) such that \( x \neq 0 \) and such that the number of nonzero terms in \( x_0 \) is as small as possible. Let
\[ x_0 = \alpha_1 e_{2k} + \alpha_2 e_{3k} + \cdots + \alpha_{p-1} e_{(p-1)k}. \]
Take \( v \in V \) such that \( \psi(k, v) \neq 0 \), and consider \( y' = (x u e_{-k}) e_{-v-k} \). It is clear that either \( y = 0 \) or \( y \) is a minimal element in \( M \), and that \( y_0 = (x_u e_{-k}) e_{-v-k} \). We have
\[ y_0 = -\psi(k, v) \sum_{i=2}^{p-1} (i - 1) i \alpha_i e_{ik}. \]
Let \( j, 2 \leq j \leq p-1 \), be such that \( \alpha_2 = \cdots = \alpha_{j-1} = 0, \alpha_j \neq 0 \), and consider \( z = -\psi(k, v) j(j-1) j x - y \). It is clear that \( \rho(z) \leq \rho(x) \), and that the number of nonzero terms in \( z_0 \) is smaller than that of \( x_0 \). Hence by the assumption on \( x \) we have \( x = 0 \). Then for any \( t \) such that \( x_t \neq 0 \), we have \( j(j-1) \equiv t(t-1) (mod \ p) \). Hence \( t = j \) or \( t = p - j + 1 \). Therefore \( x_0 \) consists of only one nonzero term or is of the form
\[ x_0 = \alpha e_{jk} + \beta e_{(p-j+1)k}, \]
where \( \alpha, \beta \) are nonzero elements in \( K \). We rule out the second possibility as follows: We may assume without loss of generality that \( 2 \leq j < (p+1)/2 \). Let again \( v \) be such that \( \psi(k, v) \neq 0 \), and consider \( y' = (x u e_{-k}) e_{-v+(j-1)k} \). Clearly \( \rho(y') \leq \rho(x) \) and \( y_0' = (x_u e_{-k}) e_{-v+(j-1)k} \). We have
\[ x_0 e_v = \psi(k, v) j \alpha e_{v+(j+1)k} + (1 - j) \beta e_{v+(j-1)k}. \]
Since \( e_w e_{-w} = 0 \) for any \( w \), we have from the above
\[ y_0' = -\psi(k, v) j(2j - 1) \alpha e_{2jk} \neq 0. \]
Thus we have proved that \( x_0 \) consists of a single nonzero term. Let \( x = e_{jk} + \sum_{u \neq 0} x_u \), where \( 2 \leq j < p-1 \). Take \( v \in V \) such that \( \psi(k, v) \neq 0 \), and consider \( x' = (x u e_{-k}) e_{-v-k} \). We have
\[ x' = - j^2 \psi(k, v) j^2 (j-1) k + \sum (x_u e_v) e_{v-k}. \]
Hence we may assume that \( j = p - 1 \); \( x = e_{-k} + \sum_{u \neq 0} x_u \). Then from the above we have \( x_0' = 0 \). Hence \( \rho(x') < \rho(x) \); \( x_0' = 0 \); \( (x_u e_v) e_{-v-k} = 0 \) for all nonzero \( u \) in \( U \). Similarly \( (x_u e_v) e_{-v} = 0 \) for all \( u \in U \). Let
Then \((x_u e_v)e_{-v} = 0\) implies
\[
\beta_i \phi(u, -v) + \beta_{i-1} \psi(u + (i - 1)k, -v) = 0
\]
for all \(i\); \((x_u e_v)e_{-v-k} = 0\) implies
\[
\beta_i \phi(u, -v) + \beta_{i-1} \psi(u + (i - 2)k, -v - k) = 0
\]
for all \(i\). Hence it follows that \(\beta_{i-1} \psi(u+v, k) = 0\) for all \(i\) whenever \(\psi(k, v) \neq 0\). Using \(2v\) instead of \(v\) we have \(\beta_{i-1} \psi(u+2v, k) = 0\). Therefore \(\beta_{i-1} = \beta_{i-1}\). Thus we have shown that \(x_u e_v = 0\) whenever \(\psi(k, v) \neq 0\). Then by proceeding as in the proof of Lemma 2.1 we may conclude that \(x_u = 0\) for all nonzero \(u\) in \(U\). Thus it is shown that the ideal \(M\) contains \(e_{-k}\).

3. The simplicity of \(L\). Let \(M\) be an arbitrary nonzero ideal of \(L\). By the preceding section \(M\) contains \(e_{-k}\), and since \(e_{-v} e_v = -\psi(k, v) e_v\), \(M\) also contains \(e_v\) for every \(v\) such that \(\psi(k, v) \neq 0\). Consider now an element \(u\) in \(V\) such that \(\psi(k, u) = 0\), \(u \neq 0\), \(u \neq k\), \(u \neq -k\).

Case I. \(\psi(u, w) = 0\) for all \(w\) in \(V\). In this case take an element \(v\) in \(V\) such that \(\psi(k, v) \neq 0\). Since \(e_{-v}\) is in \(M\), \(e_{u+v} e_{-v} = \phi(u, -v) e_u\) is in \(M\). Hence if \(e_u\) is not in \(M\), then \(\phi(u, v) = 0\) for all \(v\) such that \(\psi(k, v) \neq 0\). Let \(\psi(k, v') = 0\). Then \(\psi(k, v+v') \neq 0\); \(\psi(u, v+v') = 0\); \(\psi(u, v') = 0\). Thus \(\psi(u, w) = \phi(u, w) = 0\) for all \(w\) in \(V\). Then by (0.1) we have \(u = 0\), a contradiction. In this case, therefore, \(e_u\) is contained in \(M\).

Case II. \(\psi(u, w) \neq 0\) for some \(w\) in \(V\). In this case there exists \(v \in V\) such that \(\psi(u, v) \neq 0\), \(\psi(k, v) \neq 0\), for if \(\psi(u, v) = 0\) for all \(v\) such that \(\psi(k, v) \neq 0\), then it may be proved as in Case I that \(\psi(u, w) = 0\) for all \(w\) in \(V\). Take an arbitrary \(v\) such that \(\psi(k, v) \neq 0\). Then, since \(e_{-v-k}\) is in \(M\), \(e_{u+v} e_{-v-k}\) is also in \(M\). We have

\[
(3.1) \quad e_{u+v} e_{-v-k} = \phi(u, -v) e_{u-k} + \psi(u, -v) e_u \subseteq M.
\]

If \(u = 2k\) then from (3.1) it follows that \(e_u\) is in \(M\), since there exists \(v\) such that \(\psi(u, v) \neq 0\), \(\psi(k, v) \neq 0\). Therefore we may assume \(u \neq 2k\).

If \(e_u\) is not in \(M\) then from (3.1) it follows that there exist \(\alpha \neq 0\) and \(\beta\) in \(K\) such that

\[
(3.2) \quad \alpha \phi(u, v) + \beta \psi(u, v) = 0
\]

for all \(v\) for which \(\psi(k, v) \neq 0\). If \(\psi(k, v') = 0\) then \(\psi(k, v+v') \neq 0\), and hence we may replace \(v\) in (3.2) by \(v+v'\). From this we may infer that (3.2) is true for all \(v \in V\). Since \(u \neq 0\), (0.1) now implies that \(\beta = 0\). Then from (3.2) it follows that \(e_u \in M\), a contradiction. Thus
we have shown that every basis element is contained in $M$; $M = L$. Thus the simplicity of $L$ is proved.

4. Remarks. The algebra $L$ considered above may be generalized in the obvious way if we use $m + 1$ biadditive functions $\phi_0, \phi_1, \ldots, \phi_m$ instead of $\phi$ and $\psi$, and $m$ constant elements $k_1, \ldots, k_m$ instead of $k$ (see [2] for instance). The author, however, has been unable to obtain any result in this direction. Finally we remark that the Lie algebra $L$ and its generalizations mentioned above are all obtained as special cases of Lie algebras constructed as follows: Let $\mathfrak{A}$ be the group algebra over $K$ of an elementary $p$-group, and let $(D_1, \ldots, D_r)$ be an orthogonal system (see [3] for definition) of derivations of $\mathfrak{A}$. Let $a_1, \ldots, a_r$ in $\mathfrak{A}$ be such that $D_i a_j = D_j a_i$ for all $i, j$. Suppose that $r^2$ elements $a_{ij}$, $(i, j = 1, \ldots, r)$, in $\mathfrak{A}$ satisfy the following conditions:

$$a_{ij} + a_{ji} = 0; \quad a_{ii} = 0$$

for all $i$ and $j$;

$$\sum_{s=1}^{r} (a_{is} D_s a_{jk} + a_{js} D_s a_{ki} + a_{ks} D_s a_{ij} + a_{is} a_{jk} + a_{js} a_{ki} + a_{ks} a_{ij}) = 0,$$

for all $i, j, k$. Then an elementary computation shows that the set $L$ of derivations of $\mathfrak{A}$ of the form

$$E_f = \sum_{s, t} a_{st} (D_{sf} - a_{sf}) D_s$$

where $f$ runs over $\mathfrak{A}$, forms a subalgebra of the derivation algebra of $\mathfrak{A}$. We have $[E_f, E_g] = E_h$, where

$$h = \sum_{s, t} a_{st} (D_{sf} - a_{sf})(D_s g - a_s g).$$

The Lie algebras considered in this note and those in [2] can be obtained by taking some special values of $a_i, a_{ij}$. In case $\det (a_{ij})$ is a unit in $\mathfrak{A}$, we may simplify the conditions (4.1)–(4.2) by using $(b_{ij}) = (a_{ij})^{-1}$. It may be seen easily that (4.1)–(4.2) are equivalent to the following:

$$b_{ij} + b_{ji} = 0; \quad b_{ii} = 0; \quad (D_i b_{jk} - a_i b_{jk}) + (D_j b_{ki} - a_j b_{ki}) + (D_k b_{ij} - a_k b_{ij}) = 0$$

for all $i, j, k$. 

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A NOTE ON COMPLETE BOOLEAN ALGEBRAS

R. S. PIERCE

1. Introduction. Among commutative rings, Boolean algebras stand just below fields in simplicity of structure. In contrast, little is known concerning their classification. The purpose of this paper is to present a decomposition theorem for complete Boolean algebras, which, in a small way, simplifies the classification problem. As an illustration of how this decomposition theorem can be used, it is shown that an infinite cardinal number $\aleph$ can be the power of a complete Boolean algebra if and only if $\aleph^\aleph = \aleph$.

The words "complete Boolean algebra" will henceforth often be abbreviated C.B.A. Similarly, B.A. stands for Boolean algebra. The join, meet and complement operations in a B.A. are denoted $\lor$, $\land$ and ('). Inclusion is indicated by $\subseteq$. Also 0 and 1 respectively stand for the zero and unit elements of a B.A. The least upper bound of a subset $A$ of a C.B.A. is designated l.u.b. $A$.

The relation of isomorphism between Boolean algebras is denoted $\cong$. If $B$ is a B.A. and $a \in B$, then $B_a$ will stand for the set $\{b \in B \mid b \leq a\}$. If $a \neq 0$, then $B_a$ is a B.A. (which is complete if $B$ is complete) with the join, meet and zero inherited from $B$ and with complementation relative to $a$. That is, the complement of $b \leq a$ in $B_a$ is $b' \land a$. A Boolean algebra $B$ is called homogeneous if $B_a \cong B$ for all $a \neq 0$ in $B$. A useful property of C.B.A.'s is they satisfy the Schroeder-Bernstein theorem: if $B \cong B_a$ and $\overline{B} \cong \overline{B}_a$ ($a \in B$, $\bar{a} \in \overline{B}$), then $B \cong \overline{B}$. This fact is proved in various places, perhaps the most accessible of which is [5, Theorem 1.31].

The direct union of a set $\{B_\sigma \mid \sigma \in S\}$ of B.A.'s is defined in the usual way (see [1, p. viii]). This direct union will be denoted $\sum_{\sigma \in S} B_\sigma$. There is a useful internal characterization of direct unions of C.B.A.'s.