CONVEX SETS AND NEAREST POINTS. II

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1. Introduction. In 1935 Motzkin [5] showed that if $S$ is a subset of the euclidean plane $E$ and $z \in S$ then the set $S_z$ of all points in $E$ having $z$ as a nearest point in $S$ is closed and convex. In [7] this result was easily extended to inner product spaces of arbitrary dimension. In §2 of the present paper we show that any closed convex subset $T$ of a complete inner product space $E$ can be realized in such a manner, i.e. there exists a set $S$ and a point $z \in S$ such that $T = S_z$. Further, it is true that if $E$ is a normed linear space of dimension at least three then every closed convex subset $T$ can be so realized only if $E$ is a complete inner product space.

If $A$ is a subset of a normed linear space $E$ and $x, y$ are in $E$ we say that $y$ is point-wise closer to $A$ than is $x$ provided $\|y - a\| < \|x - a\|$ for each $a \in A$. If $x$ is such that no point of $E$ is point-wise closer to $A$ than is $x$ we call $x$ a closest-point to $A$. Fejér [1] has noted that in the euclidean plane the set $C(A)$ of all closest points to $A$ is precisely $K(A)$, the closed convex hull of $A$. In applying this result, Fejér, and later Motzkin and Schoenberg [6], actually used a weaker version, which we will call property (F): If $A \subset E$, then $C(A) \subset K(A)$. In §3 we show that property (F) characterizes complete inner product spaces of dimension at least three, while in two-dimensional spaces it is equivalent to strict convexity. We also extend Fejér's characterization of $K(A)$ to complete inner product spaces and show that it holds in strictly convex two-dimensional spaces if $A$ is bounded.

2. Nearest-point sets. If $S$ is a subset of a normed linear space $E$ and $z \in S$, $S_z$ will denote the set $\{x: \|x - z\| = \inf_{y \in S} \|x - y\|\}$ of all points in $E$ whose distance from $S$ is attained at $z$. Using the alternative formulation $S_z = \{x: \|x - z\| = \inf_{y \in S} \|x - y\|\}$ for each $y \in S$ it is easy to see that $S_z$ is always closed. In several papers [2, 3; 4] James has successfully exploited a concept of orthogonality which is defined as follows: We say that $x$ is orthogonal to $y$ (and write $x \perp y$) if $\|x\| \leq \|x - \lambda y\|$ for each $\lambda \in R$, where $R$ is the real numbers. Note that this is equivalent to saying that $x$ has the origin $\phi$ as a nearest point.

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in $Ry = \{ \lambda y : \lambda \in \mathbb{R} \}$, the line determined by $y$ and $\phi$ (assuming $y \neq \phi$)—in other words, $x \in (Ry)\phi$. That this is a generalization of the usual inner product space concept of orthogonality is easily verified, using the identity

$$\|x - \lambda y\|^2 = \|x\|^2 - 2\lambda(x, y) + \|y\|^2$$

by merely showing that in such a space $x \perp y$ if and only if $(x, y) = 0$. If $L$ is a linear subset of $E$ we write $L \perp y (y \perp L)$ provided $x \perp y (y \perp x)$ for each $x \in L$, and if $M$ is linear, $L \perp M$ is defined in a similar way. Note that $x \perp y$ if and only if $Rx \perp Ry$.

James has proved a theorem [4, Theorem 4] which, after a slight extension, will be quite useful to us.

**Theorem 1 (James).** Suppose $E$ is a normed linear space of dimension at least three. Then $E$ is a complete inner product space provided each hyperplane through the origin is orthogonal to some point $x \neq \phi$.

**Proof.** James has shown that the condition is sufficient to imply that $E$ is an inner product space, so we need only show that $E$ is also complete. Suppose $y \in F$, where $F$ is the completion of $E$. Then the set $H = \{ z : z \in F \text{ and } (z, y) = 0 \}$ is a hyperplane in $F$ and $H_0 = H \cap E$ is a hyperplane in $E$ which contains $\phi$. By hypothesis there exists $x \neq \phi$ such that $H_0 \perp x$. But, as noted previously, this implies that $(z, x) = 0$ for each $z \in H_0$. Hence, by continuity of the inner product and the fact that $H_0$ is dense in $H$, we have $(z, x) = 0$ for each $z \in H$. This implies that $y \in Rx \subset E$, so that $E = F$ is complete.

Note that if $H$ is a hyperplane in an inner product space and $x$ is a point of least norm in $H$ then $x \in (H - x)\phi$ or $x \perp H - x$.

We call a subset $T$ of a normed linear space $E$ a nearest-point set if there exists a set $S \subset E$ and a point $z \in S$ such that $T = S$. In [7] it was shown that every nearest-point set in $E$ is closed and convex if and only if $E$ is an inner product space. Here we consider the question: For what class of normed linear spaces is every closed convex set a nearest-point set? This is partially answered by the following theorem.

**Theorem 2.** In a complete inner product space $E$ every closed convex set $T$ is a nearest-point set.

**Proof.** If $T = E$, let $S = \{ \phi \}$. Otherwise, $T = \bigcap_{H \in \mathcal{H}} H'$ where $\mathcal{H}$ is the collection of all hyperplanes $H$ such that $H$ determines a closed half-space $H'$ for which $T \subset H'$. Pick any point of $T$—we will suppose that it is the origin. For $H \in \mathcal{H}$ let $x_H$ be the point of least norm in $H$ and define $S = \{ 2x_H : H \in \mathcal{H} \} \cup \{ \phi \}$. Since $x_H \perp H - x_H$ for each
$H \subseteq \mathcal{C}$, we have $H' - x_H = \{ y : (x_H, y) \leq 0 \}$ or $H' = \{ y : (x_H, y - x_H) \leq 0 \}$. But, letting $z = y - x_H$, $(z, x_H) \leq 0$ if and only if $2(z, x_H) \leq -2(z, x_H) + \|x_H\|^2$. Thus, $H' = \{ y : \|y\|^2 \leq \|y - 2x_H\|^2 \}$ and hence $T = \cap_{H \in \mathcal{C}} H' = \{ y : \|y\|^2 \leq \|y - 2x_H\|^2 \text{ for each } H \in \mathcal{C} \} = \{ y : \|y\|^2 \leq \|y - z\|^2 \text{ for each } z \in \mathcal{S} \} = S_{\phi}$.

The above proof shows that for any $z \in T$ there exists a set $S(T, z)$ such that $T = S(T, z)$. Applying the (constructive) method of the proof to a specific example we find, with the aid of some elementary analytic geometry, that if $T$ is the closed disc bounded by the curve $r = 2 \cos \theta$ then $S(T, \phi)$ is the area not inside the cardioid $r = 2(1 + \cos \theta)$. Note that we could omit from $S$ any point not on the cardioid and still have $T = S_{\phi}$. We can, however, show that in a certain sense the set $S$ can be taken to be unique. Suppose that $T$ is a nearest-point subset of a normed linear space $E$; then there exists $S \subseteq E$ and $z \in S$ such that $T = S_z$. Let $\mathcal{S}$ be the collection of all sets $S$ containing $z$ such that $S_z = T$. Then, letting $Q(T, z) = \bigcup_{S \in \mathcal{S}} S$, we see that $Q(T, z) = \cap_{S \in \mathcal{S}} S = T$ so that $Q(T, z) \subseteq \mathcal{S}$ and is the biggest member of $\mathcal{S}$. Henceforth, if a set $T$ is a nearest-point set, $Q(T, z)$ will denote the set defined above. If there is no chance for confusion we will simply denote it by $Q$.

Suppose that $T = S_z$ for some $S$ and $z \in S$. Then $Q(T, z) = \{ x : \|x - y\| \geq \|z - y\| \text{ for each } y \in T \}$. For clearly $x \in Q$ implies $\|x - y\| \geq \|z - y\|$ for each $y \in T$; and if this latter is true, $T \subseteq Q \cup \{ x \} \subseteq Q = T$ so that $Q \cup \{ x \} = T$ and by the maximality property of $Q$, $x \in Q \cup \{ x \} \subseteq Q$. Using this description of $Q$ it is easy to see that $Q$ is closed. Further, the set $S$ constructed in the proof of Theorem 2 is actually equal to $Q(T, \phi)$, for $T = S_{\phi}$ implies that $S \subseteq \{ x : \|x - y\| \geq \|y\| \text{ for each } y \in T \}$. Suppose that $x$ is a point of the latter set, then the hyperplane $H = \{ w : \|w - x\| = \|w\| \}$ passes through $(1/2)x$ and is orthogonal to $Rx$. Hence, if $H'$ is the closed half-space determined by $H$ which contains $\phi$, $y \in T$ implies $\|y\| \leq \|y - x\|$ so that $y \in H'$ and thus $T \subseteq H'$. We have, then, that $H \in \mathcal{C}$ and that $x$ is of the form $2x_H$ and is therefore in $S$.

It is natural to wonder if $Q$ may be convex. Suppose that $T = Q$ and that $Q$ is convex, then by Lemmas 3.1 and 4.1 of [7] $T$ must be a convex cone with vertex $\phi$. Using identity (1) it is easily verified that whenever $T$ is a cone with vertex $\phi$, $Q = \{ x : (x, y) \leq 0 \text{ for each } y \in T \}$ and hence $Q$ is also a convex cone with vertex $\phi$ (the dual or polar cone of $T$ [8]). Thus, $Q$ is convex if and only if both $T$ and $Q$ are convex cones with common vertex.

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In what follows we show that the hypothesis in Theorem 2 that \( E \) be a complete inner product space is necessary, if the dimension of \( E \) is at least three.

**Theorem 3.** Suppose that \( E \) is a normed linear space of three or more dimensions and that every closed convex subset of \( E \) is a nearest-point set. Then \( E \) is a complete inner product space.\(^3\)

**Proof.** Suppose \( H \) is a hyperplane in \( E \) which passes through the origin. By Theorem 1 we can conclude that \( E \) is a complete inner product space provided every such hyperplane is orthogonal to some \( x \neq 0 \), i.e. provided \( H \perp Rx \) for some \( x \neq 0 \). Now, \( H \) is closed and convex so there exists a set \( S \) and a point \( z \in S \) such that \( H = S_z \). By the preceding discussion there exists a maximal set \( Q(H, z) \) such that \( H = Q_z \).

If \( Q = \{z\} \), \( Q_z = H \) would be the entire space, contradicting the fact that \( H \) is a hyperplane. Pick \( x \in Q \sim \{z\} \) and suppose \( y \in H \) and \( x \neq 0 \). Since \( z \in H \), there exists \( w \in H \) such that \( y = \lambda(w - z) \). We know that \( \|w - x\| \geq \|w - z\| \) hence \( \|\lambda(w - x)\| \geq \|\lambda(w - z)\| \) or \( \|y - \lambda(x - z)\| \geq \|y\| \). Therefore \( H \perp R(x - z) \), which was to be shown.

What if \( E \) is two-dimensional and every closed convex subset of \( E \) is a nearest-point set? By somewhat lengthy arguments we have been able to show that this implies strict convexity of \( E \), and that \( E \) is an inner product space provided we assume that \( E \) is smooth. We have been unable to show that the assumption of smoothness is necessary. This result indicates, however, that strict convexity alone does not imply that every closed convex subset of \( E \) is a nearest-point set, for such an implication would in turn imply the false statement that every smooth and strictly convex two-dimensional normed linear space is an inner product space.

3. **Convex sets and closest points.** We first prove Fejér's theorem (see introduction) in a more general setting.

**Theorem 4 (Fejér).** If \( A \) is a subset of a complete inner product space \( E \), the closed convex hull \( K(A) \) of \( A \) is the set \( C(A) \) of closest points to \( A \).

**Proof.** Suppose \( x \in K(A) \), then there exists a hyperplane \( H \) separating \( x \) from \( K(A) \) such that \( x \in H \). Since \( E \) is complete we know that the distance from \( x \) to \( H \) is attained at some (unique) point of \( H \). Without loss of generality we may assume that this point is the origin, so that \( \|x\| = \|x - z\| \) for each \( z \in H \). This says, then, that \( x \perp H \) and hence \( H \perp x \) Thus, if \( a \in A \), there exists \( z \in H \cap [a, x] \). Now, \( z \in K(A) \), and therefore \( \|x - z\| \leq \|x - a\| \) for each \( a \in A \).

\[^3\] We wish to thank the referee for suggesting changes which simplified the proof of this theorem, as well as the proof of Theorem 5.
\[ \|z\| < \|z-x\| \] and therefore \[ \|a\| \leq \|a-z\| + \|z\| < \|a-z\| + \|z-x\| = \|a-x\|. \] Hence \( \phi \) is point-wise closer to \( A \) than is \( x \), giving \( x \notin C(A) \).

Suppose \( x \in C(A) \), then there exists \( y \in E \) which is point-wise closer to \( A \) than is \( x \). The set \( H = \{ z : \|z-y\| = \|z-x\| \} \) is a hyperplane through \((1/2)(x+y)\) which determines a closed half-space \( H' = \{ z : \|z-y\| \leq \|z-x\| \} \) containing \( A \). Hence \( K(A) \subset H' \) and since \( x \in H' \), \( x \in K(A) \).

Defining property (F) as in the introduction, we now determine the class of spaces possessing the property.

**Theorem 5.** Suppose that \( E \) is a normed linear space of dimension at least three which possess property (F). Then \( E \) is a complete inner product space.

**Proof.** We again make use of Theorem 1. Suppose that \( H \) is a hyperplane through \( \phi \). If \( x \in H = K(H) \) then by assumption \( x \in C(H) \) and there must exist a point \( y \) such that \( \|h-y\| < \|h-x\| \) for each \( h \in H \). The line \( L \) determined by \( x \) and \( y \) must hit \( H \) at \( w \), say. (For \( \phi \) has a nearest point \( z \) in \( L \), so if \( L \) were parallel to \( H \), \( h = x-z \) would be in \( H \) and have \( x \) as a nearest-point in \( L \). In particular, \( h \) would not be closer to \( y \in L \) than to \( x \), a contradiction.) Now, \( w = \alpha y + (1-\alpha)x \) for some \( \alpha \neq 0 \), so if \( \lambda \neq 0 \) and \( h \) is any point of \( H \) we have \( \alpha \lambda^{-1}h + w \in H \) and \( \|h - \lambda(x-y)\| = \|\lambda^{-1}(\alpha \lambda^{-1}h + w) - x\| \)
\[ \|\lambda^{-1}(\alpha \lambda^{-1}h + w) - y\| = \|h + \beta \lambda(x-y)\|, \]
where we have let \( \beta = \lambda^{-1}(1-\alpha) \). If \( \beta = 0 \) we have the inequality we are seeking; otherwise, we use the fact that \( h \in H \) if and only if \( \beta^{-1}h \in H \) to conclude, by induction, that \( \|h - \lambda(x-y)\| > \|h + \beta \lambda(x-y)\| \) for each positive integer \( n \). Setting \( h = \phi \) shows that \( \|h\| < 1 \), so, taking the limit as \( n \to \infty \), we have \( \|h - \lambda(x-y)\| > \|h\| \). This shows that \( H \perp (x-y) \) and hence we can conclude from Theorem 1 that \( E \) is a complete inner product space.

**Lemma 6.** If a normed linear space \( E \) has property (F) then \( E \) is strictly convex.

**Proof.** If \( E \) is not strictly convex there exist points \( x, y \) in \( E \) such that the distance from \( y \) to each point of the segment \([-x, x]\) is exactly 1. Let \( A = [-y, y]; \) then \( x \in A = K(A) \) and hence \( x \in C(A) \), by hypothesis. There must exist a point \( z \) in \( E \) which is point-wise closer to \( A \) than is \( x \). Thus, \( \|z-y\| < \|x-y\| = 1 \) and \( \|z+y\| < \|x+y\| = 1 \) so \( 2 = \|2y\| = \|(-y)-y\| \leq \|y-z\| + \|z-y\| < 2 \), a contradiction.

**Theorem 7.** Suppose that \( E \) is a two-dimensional normed linear space. Then \( E \) possesses property (F) if and only if it is strictly convex.

**Proof.** Lemma 6 proves the necessity. Suppose \( E \) is strictly con-
vex and \( A \subseteq E \). If \( x \in K(A) \) there exists a line \( L \) separating \( K(A) \) from \( x \) such that \( x \in L \). We can suppose that \( \phi \in L \) and that \( L = Ry \) for some \( y \neq \phi \). There exists a line \( M \) through \( \phi \) such that \( y \perp M \) \cite{3, Theorem 2.2}, so \( M + x \) hits \( Ry \) at \( \alpha y \) for some \( \alpha \in R \). We will show that \( \alpha y \) is point-wise closer to \( A \) than is \( x \). Suppose \( a \in A \), then \( [a, x] \) hits \( Ry \) at \( Py \), say. Now \( M + x = M + \alpha y \) so \( x - \alpha y \in M \sim \{ \phi \} \) and hence, by strict convexity, \( ||\beta y - \alpha y|| < ||(\beta y - \alpha y) - (x - \alpha y)|| = ||\beta y - x||. \) Then

\[
||a - \alpha y|| \leq ||a - \beta y|| + ||\beta y - \alpha y|| < ||a - \beta y|| + ||\beta y - x|| = ||a - x||. 
\]

Hence \( x \in C(A) \) and we have \( C(A) \subseteq K(A) \).

In stating his theorem Fejér assumed that \( A \) was compact. This is an unnecessary restriction if we wish to prove his characterization of \( K(A) \) in the euclidean plane, but it is interesting to note that if we merely assume that \( A \) is bounded we can prove his characterization in any two-dimensional strictly convex normed linear space.

**Theorem 8.** Suppose that \( A \) is a bounded subset of a strictly convex two-dimensional normed linear space \( E \). Then \( K(A) = C(A) \).

**Proof.** That \( C(A) \subseteq K(A) \) follows from the strict convexity and Theorem 7. Suppose that \( x \in C(A) \), then there exists a point \( y \in E \) such that \( ||y - a|| < ||x - a|| \) for each \( a \in A \). We can suppose without loss of generality that \( x = \phi \). Now there exists a point \( z \neq \phi \) such that \( z \perp y \) \cite{3} and hence \( L \perp y \) for \( L = Rz \). Letting \( L' \) be the closed half-space determined by \( L \) which does not contain \( y \), we see that \( \text{cl } A \), the closure of \( A \), is contained in \( E \sim L' \) as follows: Let \( S = \{ w : ||w - y|| \leq ||w|| \} \); then \( S \) is closed, contains \( A \) and hence contains \( \text{cl } A \). Furthermore, if \( w \) is a point of \( L' \), we can use strict convexity and the triangle inequality (as in the proof of Theorem 7) to show that \( ||w|| < ||w - y||. \) Thus, \( w \in S \) so \( S \subseteq E \sim L' \) and \( \text{cl } A \subseteq E \sim L' \).

Since \( A \) is bounded, \( \text{cl } A \) is compact and hence \( \text{conv } \text{cl } A \), the convex hull of \( \text{cl } A \), is closed. Thus, \( K(\text{cl } A) = \text{conv } \text{cl } A \) and since \( E \sim L' \) is convex, \( \text{conv } \text{cl } A \subseteq E \sim L' \). We have, then, that \( K(A) \subseteq K(\text{cl } A) \subseteq E \sim L' \), so that \( \phi \), being in \( L \subseteq L' \), is not in \( K(A) \). Therefore \( C(A) = K(A) \) for any bounded set \( A \).

It is possible to construct an example in which the above result fails for an unbounded set \( A \).

**Bibliography**

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NOTE ON PRODUCTS IN Ext

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The objective of this note is to present an interrelation between the \( V \)-product in [1] and the composition product in [2; 3], which in turn gives a comparison of the cup-product with the composition product. Similar relations can be obtained also for other products involving Tor and (iterated) connecting homomorphisms.

We retain the notations in [1, Chapter XI]. The external product

\[ V: \text{Ext}_{\Lambda \otimes \Sigma^*}(A, C) \otimes \text{Ext}_{\Sigma \otimes \Gamma^*}(A', C') \rightarrow \text{Ext}_{\Lambda \otimes \Gamma^*}(A \otimes \Sigma A', C \otimes \Sigma C') \]

is defined in the situation \((\Lambda \otimes \Sigma, \Lambda C \Sigma, \Sigma A^\Gamma, \Sigma C^\Gamma)\) under the following assumption: (i) \( \Lambda, \Gamma, \Sigma \) are \( K \)-projective; (ii) \( \text{Tor}_{n}^\Sigma(A, A') = 0 \) for \( n > 0 \). To this situation we now add \((\Lambda B \Sigma, \Sigma B^\Gamma)\) and (ii') \( \text{Tor}_{n}^\Sigma(B, B') = 0 \) for \( n > 0 \). For \( a \in \text{Ext}_{\Lambda \otimes \Sigma^*}(A, B) \) and \( b \in \text{Ext}_{\Sigma \otimes \Gamma^*}(B, C) \) the composition product \( b \circ a \) lies in \( \text{Ext}_{\Lambda \otimes \Sigma^*}(A, C) \). For \( a' \in \text{Ext}_{\Sigma \otimes \Gamma^*}(A', B') \) and \( b' \in \text{Ext}_{\Lambda \otimes \Sigma^*}(B', C') \), \( b' \circ a' \) lies in \( \text{Ext}_{\Sigma \otimes \Gamma^*}(A', C') \). \( (b \circ a = a \circ b \) in the notation of [2].)

**Proposition 1.** \((b \circ a) V(b' \circ a') = (-1)^{p'q'}(bVb') \circ (aVa')\)

In fact let \( X, Y \) be \( \Lambda \otimes \Sigma^* \)-projective resolutions of \( A, B \) respectively, and let \( X', Y' \) be \( \Sigma \otimes \Gamma^* \)-projective resolutions of \( A', B' \). Then \( X \otimes \Sigma X', Y \otimes \Sigma Y' \) are \( \Lambda \otimes \Gamma^* \)-projective resolutions of \( A \otimes \Sigma A', B \otimes \Sigma B' \). We consider these resolutions as chain complexes with 0's in negative dimensions. Suppose that \( a, b, a', b' \) are respectively represented by maps \( \alpha: X_p \rightarrow B, \beta: Y_q \rightarrow C, \alpha': X'_p \rightarrow B', \text{ and } \beta': Y'_q \rightarrow C' \). The map \( \alpha \) Received by the editors April 25, 1958.

\(^1\) Work done while the author was engaged under contract NONR 2383.