A SEMI-SIMPLE MATRIX GROUP IS OF TYPE I
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The purpose of the following note is to give a short simple proof of the most important special case of Harish-Chandra's result [3, Theorem 7] that every connected semi-simple Lie group is of type I. We shall prove that every continuous unitary representation of a connected semi-simple matrix group is of type I. This fact is a consequence of theorems in two papers of Godement [1, Theorem 2; 2, Theorem 8]. An improved version of Godement's method is used here. The improvements are:

1. the argument is considerably shorter;
2. infinite dimensional nonunitary representations are not needed;
3. the argument is completely global; no direct integrals are used.

These improvements seem not to have been noticed before.

Definition. We shall say that an algebra is of type $I_{\infty}$ if it satisfies the identities

$$[A_1, \ldots, A_r] = \sum_{i_1, \ldots, i_r} \operatorname{sgn}\left(\begin{array}{c} 1 \cdots r \\ i_1 \cdots i_r \end{array}\right) A_{i_1} A_{i_2} \cdots A_{i_r} = 0$$

for all $r$ for which the algebra of all $n \times n$ matrices satisfies them.

In [5, §2], Kaplansky shows that the algebra of all $n \times n$ matrices is of type $I_{\infty}$ but not of type $I_{\infty-1}$. It follows from the definition that an algebra is of type $I_{\infty}$ if it is a subalgebra of an algebra of type $I_{\infty}$, or if it is a homomorphic image of an algebra of type $I_{\infty}$, or if it has a separating family of homomorphisms into algebras of type $I_{\infty}$. Since the above identities are linear in each variable, a von Neumann algebra is of type $I_{\infty}$ if it has a weakly dense subalgebra of type $I_{\infty}$. A von Neumann algebra of type I (in the usual sense) is
the direct sum of von Neumann algebras of homogeneous type $I_k$ where $k$ runs over the cardinals. Since an algebra of type $I_{\mathbb{R}^n}$ in the sense of the above definition has no subalgebras isomorphic to the $(n+1)\times(n+1)$ matrices, it is clear that a von Neumann algebra is of type $I_{\mathbb{R}^n}$ as defined above if and only if it is of type $I$ in the usual sense and all the direct summands of homogeneous type $I_k$ are zero for $k>n$. Consequently the present terminology accords with the usual one.

**Theorem 1.** Let $G$ be a connected semi-simple matrix group. Let $L$ be the left regular representation of $G$ on $L_2(G)$, and let $\mathcal{L}$ be the von Neumann algebra generated by $L$. Let $K$ be a maximal compact subgroup of $G$, and let $\rho$ be an irreducible representation of $K$ of degree $n$ with character $\chi/n$. Let $E$ be the projection given by

$$E = \int_K \chi(k)L(k)dKk$$

where $dKk$ is Haar measure on $K$ and the integral is in the weak sense. Then the algebra $E\mathcal{L}E$ is of type $I_{\mathbb{R}^2}$.

**Proof.** The group $G$ has a connected solvable subgroup $S$ such that $G = KS$ (see [4, Lemma 3.11 and the proof of Lemma 3.12]). Let $\alpha$ be the algebra of all $L_\phi$ with $\phi$ in $C_0(G)$ where $L_\phi = \int_G \phi(x)L(x)dx$. It is sufficient to show that $E\alpha E$ is of type $I_{\mathbb{R}^2}$. This will be done by exhibiting a separating family of homomorphisms of $E\alpha E$ into $q\times q$ matrices with $q \leq n^2$. Such a family is provided by the finite dimensional irreducible representations of $G$.

If $\gamma$ is a finite dimensional representation of $G$ we consider the homomorphism $L_{\gamma} \rightarrow \int f(x)\gamma(x)dx$ of $\alpha$ into operators on the representation space of $\gamma$. Since $G$ has a faithful finite dimensional representation, the Stone-Weierstrass theorem implies that these homomorphisms separate $\alpha$. Consequently, since $G$ is semi-simple, the ones arising from irreducible $\gamma$ separate $\alpha$. Let $Q = \int_K \chi(k)\gamma(k)dKk$. Then $E\mathcal{L}E \rightarrow Q(\int f(x)\gamma(x)dx)Q$ under the homomorphism. It is, therefore, sufficient to observe that the dimension $q$ of the range of the projection $Q$ is $\leq n^2$. But this is a consequence of the fact that $\gamma|K$ is a cyclic representation, which we shall now show. By Lie's theorem there exists a vector $v$ such that $\gamma(s)v = \lambda(s)v$ for all $s$ in $S$ where $\lambda(s)$ is a scalar. Since $\gamma$ is irreducible $\gamma(G)v = \gamma(K)\gamma(S)v = \lambda(S)\gamma(K)v$ spans the representation space, and so $\gamma(K)v$ does also.

**Corollary.** Let $G$ be a connected semi-simple matrix group. Then every continuous unitary representation of $G$ is of type $I$. 

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Proof. Let $U$ be a unitary representation of $G$, and let $\rho$ be any irreducible representation of a maximal compact subgroup $K$. Let $\tilde{\chi}_\rho / n$ be the character of $\rho$ where $n$ is the degree of $\rho$. Let $P_\rho = \int \chi_\rho(x) U(x) dx$. Let $\mathfrak{u}$ be the von Neumann algebra generated by $U$. Then $P_\rho \mathfrak{u} P_\rho$ is of type $I_\frac{1}{n}$ since a number of subalgebras of $E \subset E$ have homomorphic images which are weakly dense in $P_\rho \mathfrak{u} P_\rho$. But $\{ P_\rho \}$ is an orthogonal family of projections whose sum is $I$. Hence $\mathfrak{u}$ is of type $I$.

Bibliography


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