APPROXIMATION BY A POLYNOMIAL AND ITS DERIVATIVES ON CERTAIN CLOSED SETS

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The work on the theory of approximations initiated by Weierstrass and continued by Walsh, Keldysh, and Lavrentiev, among others, has culminated in the following theorem of Mergelyan (See Mergelyan [3]): Given any compact subset $C$ of the complex plane, which does not separate the plane, and given any continuous function $f$ on $C$ which is analytic interior to $C$, then $f$ can be approximated uniformly on $C$ by polynomials.

This theorem leaves the following question unanswered: If $f_0, f_1, \ldots, f_n$ are continuous functions on $C$, can a sequence $\{p_k\}$ of polynomials be found with the property that for each integer $i$ with $0 \leq i \leq n$ the sequence $\{p_k^{(i)}\}$, where $p_k^{(i)}$ denotes the $i$th derivative of $p_k$, converges uniformly on $C$ to $f_i$? If $C$ is totally disconnected, it is easy to show that the answer to this question is always yes. We omit the simple proof, because a more general result will be given elsewhere. If $C$ is a Jordan arc, the question becomes more complicated. It is clear that if $C$ has a rectifiable sub-arc $J$, whose endpoints we call $z_0$ and $z_1$, then for the approximation to be possible it is necessary that $\int_{f_i+1}(z)dz = f_i(z_1) - f_i(z_0)$ for $0 \leq i \leq n - 1$. Thus, if the approximation is to be possible whatever the functions $f_0, f_1, \ldots, f_n$, it is necessary that $C$ have no rectifiable sub-arcs. Conversely, if $C$ is a Jordan arc having no rectifiable sub-arcs, we conjecture that the approximation is always possible. It is the purpose of this paper to prove this conjecture by means of an additional hypothesis, that $C$ satisfy a Lipschitz condition of a fixed order $c$ at a dense set of points. (This concept will be defined below.) The author has been unable to prove the conjecture without this restriction.

If $S_1$ and $S_2$ are any subsets of the complex plane, define $d(S_1, S_2) = \min \{|z_1 - z_2| \mid z_1 \in S_1, z_2 \in S_2\}$.

**Definition 1.** Let $\phi$ be a homeomorphic map of $[0, 1]$ into the complex plane, so that $\phi[0, 1]$ is a Jordan arc $C$. We say that $C$ satisfies a Lipschitz condition of order $c$ at a point $\phi(t)$ of $C$, $t \in [0, 1]$, if there exist $A > 0$ and $\delta > 0$ such that max $\{d(\phi[0, t], z), d(\phi[t, 1], z)\} \geq A|\phi(t) - z|^c$ whenever $|\phi(t) - z| < \delta$.

Then we have

**Theorem 1.** If $C$ has no rectifiable sub-arcs and if there exists $c > 0$ such that $C$ satisfies a Lipschitz condition of order $c$ at a dense set $S$ of

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points, then for any continuous functions \( f_0, \ldots, f_n \) on \( C \) there exists a sequence \( \{p_i\} \) of polynomials for which \( p_i^{[k]} \to f_k \) uniformly on \( C \) as \( i \to \infty \), for \( 0 \leq k \leq n \).

The proof of Theorem 1 will utilize the Riesz representation theorem (see Banach [1, p. 60]). If \( X \) is the set of all \( n+1 \) tuples \( (f_0, \ldots, f_n) \) of continuous functions on \( C \), topologized by the norm \( \| (f_0, \ldots, f_n) \| = \sup \{ |f_i(z)| : z \in C, 0 \leq i \leq n \} \), then by the Riesz theorem we see that to any bounded linear functional \( L \) on \( X \) correspond unique measures \( \mu_0, \ldots, \mu_n \) on \([0, 1]\) such that \( L(f_0, \ldots, f_n) = \int_0^1 f_0(\phi(t))d\mu_0(t) + \cdots + \int_0^1 f_n(\phi(t))d\mu_n(t) \). Now let \( Y \) be the subset of \( X \) consisting of all \( (\phi, p^{[1]}, \ldots, p^{[n]}) \), where \( \phi \) is any polynomial. Then Theorem 1 states that \( Y \) is dense in \( X \). By the Hahn-Banach theorem (see Banach [1]), this is equivalent to saying that every bounded linear functional on \( X \) which vanishes on \( Y \) vanishes on \( X \).

If \( L \) is the bounded linear functional in question, then by the above representation of \( L \) we see that \( L(p, \ldots, p^{[n]}) = \int_0^1 p(\phi(t))d\mu_0(t) + \cdots + \int_0^1 p^{[n]}(\phi(t))d\mu_n(t) = 0 \) for all polynomials \( p \). To prove Theorem 1 we must show that this implies that \( \mu_0 = \mu_1 = \cdots = \mu_n = 0 \).

The linear functional \( L \) and therefore \( \mu_0, \ldots, \mu_n \) will be fixed during the discussion. As an abbreviation we set \( L_t(p) = \int_0^1 p(\phi(t))d\mu_0(t) + \cdots + \int_0^1 p^{[n]}(\phi(t))d\mu_n(t) \) for all \( t \) in \([0, 1]\) and all functions \( p \) analytic in some neighborhood of \( C \). Then \( L_t(p) = 0 \).

Assume now that \( \phi(t) \in S \). We proceed to obtain a new formula for \( L_t(p) \). To do this, take \( A \) and \( \delta \) as in Definition 1, and let \( \epsilon \) be positive and less than \( \delta \). Let \( J \) be the circle \( |z - \phi(t)| = \epsilon \). Then by Definition 1, for \( z \) in \( J \) either \( d(z, U_1) > A\epsilon^e \) or \( d(z, U_2) > A\epsilon^e \), where \( U_1 = \phi[0, t] \) and \( U_2 = \phi[t, 1] \). Thus we can write \( J = J_1 \cup J_2 \), where \( J_1 \) and \( J_2 \) are disjoint Borel sets and \( d(z, U_i) > A\epsilon^e \) for \( z \) in \( J_i \). Now if \( p \) is any polynomial, let \( K = \max \{ |p(z)| : z \in J \} \). Then

\[
p(z) = (1/2\pi i) \int_J p(\xi) d\xi/(\xi - z)
\]

for \( |z - \phi(t)| < \epsilon \). If for \( i = 1 \) and \( 2 \) we define

\[
f_i(z) = (1/2\pi i) \int_{J_i} p(\xi) d\xi/(\xi - z),
\]

then we see that \( f_i \) is analytic on the complement of the closure of \( J_i \), that \( |f_i^{[l]}(z)| \leq K_j^l d(z, J_i)^{-(l+1)} \), and that \( p(z) = f_1(z) + f_2(z) \) for \( |z - \phi(t)| < \epsilon \). Thus we see that \( |f_i^{[l]}(z)| < K_j^l[A\epsilon^e]^{-(l+1)} \) for \( z \) in \( U_i \).

Now let \( 0_1 \) be a neighborhood of \( \phi[0, t] \) and \( 0_2 \) a neighborhood of \( \phi[t, 1] \) such that \( 0_1 \cap 0_2 = \{ z \mid |z - \phi(t)| < \epsilon \} \) and \( 0_1 \cap J_1 = 0_2 \cap J_2 = \phi \).

Then \( f_1 + f_2 = p \) on \( 0_1 \cap 0_2 \) and \( f_i \) is analytic in \( 0_i \). Therefore we may
define an analytic function $g_1$ on $O_1 \cup O_2$ by specifying $g_1(z) = f_1(z)$ for $z$ in $O_1$ and $g_1(z) = p(z) - f_2(z)$ for $z$ in $O_2$. Also define $g_2$ on $O_1 \cup O_2$ by $g_2(z) = f_2(z)$ for $z$ in $O_2$ and $g_2(z) = p(z) - f_1(z)$ for $z$ in $O_1$. Then $g_1 + g_2 = p$ in $O_1 \cup O_2$. Also $|g_1^{[j]}(z)| = |f_1^{[j]}(z)| \leq K_j! [A e^e]^{-j+1}$ for $z$ in $U_i$, where $i = 1$ or $2$ and $j$ is arbitrary. For $i = 1$ this inequality, in conjunction with the definition of $L_i(g_1)$, tells us that $|L_i(g_1)| \leq M_1 K e^{-c(n+1)}$, where $M_1$ is a constant. For $i = 2$ the inequality tells us that $|L_i(g_2)| \leq M_2 K e^{-c(n+1)}$. Thus we see that $|L_i(p)| = |L_i(g_1) + L_i(g_2)| \leq |L_i(g_1)| + |L_i(g_2)| = |L_i(g_1)| + |L_i(g_2) - L_i(g_2)| \leq M K e^{-m}$, where $M = M_1 + M_2$ and $m$ is any integer larger than $c(n+1)$. Taking $p$ to be the polynomial $(z - \phi(t))^i$, we see that $K = e^i$, so that $L_i([z - \phi(t)]^i) < M e^{-m}$ for all $e < \delta$. If $j > m$, this implies that $L_i([z - \phi(t)]^i) = 0$. Therefore $L_i(p)$ depends only on the first $m + 1$ terms of the expansion of $p$ in powers on $z - \phi(t)$, so that

$$L_i(p) = \sum_{i=0}^{m} \beta_i(\ell) p^{[i]}(\phi(\ell)), \text{ where } \beta_0(\ell), \ldots, \beta_m(\ell)$$

are certain complex numbers.

For the remainder of the proof, the only use which will be made of the fact that $C$ satisfies a Lipschitz condition of order $c$ at points of $S$ will be to conclude that the expression just obtained for $L_i(p)$ is valid whenever $\phi(t)$ is in $S$. Therefore, the conjecture of the introductory paragraphs can be proved whenever the expression just obtained for $L_i(p)$ can be shown to be valid for a set of values of $t$ which is dense in $[0, 1]$.

We now obtain another formula for $L_i(p)$, where now $t$ may be any point in $(0, 1)$. For any polynomial $p$ and any complex number $z$ we have the Taylor's formula

$$p(z) = \sum_{i=0}^{\infty} p^{[i]}(\phi(\ell)) \frac{[z - \phi(\ell)]^i}{i!}.$$ 

Thus

$$L_i(p) = \sum_{j=0}^{n} \int_{0}^{t} \left\{ \sum_{i=0}^{\infty} p^{[i+j]}(\phi(\ell)) \frac{[\phi(x) - \phi(\ell)]^i}{i!} \right\} d\mu_j(x)$$

$$= \sum_{i=0}^{\infty} p^{[i]}(\phi(\ell)) \left( \sum_{j=0}^{i} \int_{0}^{t} \frac{[\phi(x) - \phi(\ell)]^{i-j}}{(i-j)!} d\mu_j(x) \right),$$

where $\mu_j = 0$ if $j > n$. If we define

$$\alpha_i(t) = \sum_{j=0}^{i} \int_{0}^{t} \frac{[\phi(x) - \phi(\ell)]^{i-j}}{(i-j)!} d\mu_j(x),$$

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for \( t \) in \((0, 1)\), we therefore have \( L_t(p) = \sum_{i=0}^{m} \beta_i(t) p^{[i]}(\phi(t)) \alpha_i(t) \). We see that \( \alpha_i \) is continuous on the right. Comparing the two formulas obtained for \( L_t(p) \), we see that 
\[
\sum_{i=0}^{m} \beta_i(t) p^{[i]}(\phi(t)) = \sum_{i=0}^{m} \alpha_i(t) p^{[i]}(\phi(t))
\]
for all \( t \) in \((0, 1)\) with \( \phi(t) \in S \), and for all polynomials \( p \). It follows that \( \alpha_i(t) = 0 \) for \( i > m \), \( t \in (0, 1) \), and \( \phi(t) \in S \). Since \( S \) is dense and since \( \alpha_i \) is continuous on the right, it follows that \( \alpha_i = 0 \) for \( i > m \). Before proceeding, we need a definition.

**Definition 2.** If \( f \) and \( g \) are two complex valued functions on \([0, 1]\), if \( t \in [0, 1) \), and if \( a \) is a complex number, then \( a \) is said to be a right conditional derivative at the point \( t \) of \( f \) with respect to \( g \) if 
\[
\lim_{i \to \infty} \left\{ \frac{[\phi(t) - \phi(t_0)]}{[\alpha_i(t) - \phi(t_0)]} \right\} [\phi(t) - \phi(t_0)]^{-1} d\mu_j(x)
\]
for all \( i \) in \((0, 1)\) with \( \phi(t) \in S \), and for all polynomials \( p \). It follows that \( \alpha_i(t) = 0 \) for \( i > m \), \( t \in (0, 1) \), and \( \phi(t) \in S \). Since \( S \) is dense and since \( \alpha_i \) is continuous on the right, it follows that \( \alpha_i = 0 \) for \( i > m \). Before proceeding, we need a definition.

To get more information about \( \alpha_i \), take \( 0 < t_0 < t < 1 \), set \( \lambda_i(t) = \mu_i[0, t] = \int_0^t d\mu_j(x) \), and consider the difference quotient
\[
\sum_{j=0}^{i-1} \int_0^t \frac{[\phi(x) - \phi(t_0)]}{(i - j)!} d\mu_j(x)
\]
for \( \phi(t) - \phi(t_0) = \phi(x) - \phi(t) \) for all \( x \) in \([0, t]\). The latter condition will be satisfied if \( \phi(t) - \phi(t_0) = \phi(x) - \phi(t_0) \) for all \( x \) in \([t_0, t]\), and values of \( t \) can be found arbitrarily close to \( t_0 \) for which this will be true. Thus the second summation can be made arbitrarily small for certain values of \( t \) close to \( t_0 \). Due to uniform convergence under the integral signs, the other of the above summations approaches
\[
\sum_{j=0}^{i-1} \int_0^{t_0} \frac{\phi(x) - \phi(t_0)}{(i - j - 1)!} d\mu_j(x) = -\alpha_{i-1}(t_0) \text{ as } t \to t_0.
\]
Thus we see that \( \alpha_{i-1}(t_0) \) is a right conditional derivative at \( t_0 \) of \( \lambda_j - \alpha_j \) with respect to \( \phi \), for \( j \geq 1 \) and all \( t_0 \) in \((0, 1)\).
Since $\alpha_j = 0$ for $j > m$ and since $\lambda_j = 0$ for $j > n$, if $m > n$ we see that $\alpha_m(t)$, which is a right conditional derivative at $t$ of $\lambda_{m+1} - \alpha_{m+1} = 0$ with respect to $\phi$, must vanish for $t$ in $(0, 1)$. Thus $\alpha_m = 0$. The argument can then be continued to show step by step that $\alpha_i = 0$ for $i \geq n$.

Therefore $\alpha_{n-1}(t)$ is a right conditional derivative at the point $t$ of $\lambda_n - \alpha_n = \lambda_n$ with respect to $\phi$, for all $t$ in $(0, 1)$. If $\alpha_{n-1}(t) \neq 0$, this implies that $[\alpha_{n-1}(t)]^{-1}$ is a right conditional derivative at $t$ of $\phi$ with respect to $\lambda_n$. Since $\alpha_{n-1}$ is continuous on the right, we can find $u > t$ and $r > 0$ such that $|\alpha_{n-1}(x)| > r$ for $x$ in $[t, u)$. Therefore for $x$ in $[t, u)$ we see that $|\alpha_{n-1}(x)|^{-1} < r^{-1}$ and $[\alpha_{n-1}(x)]^{-1}$ is a right conditional derivative at $x$ of $\phi$ with respect to $\lambda_n$. Hence there exist points $x'$ arbitrarily close to $x$ on the right with $|[\phi(x') - \phi(x)] \cdot [\lambda_n(x') - \lambda_n(x)]^{-1}| < r^{-1}$. Given any $x$ and $y$ in $[t, u)$, $x < y$, let $T$ be the set of all $x'$ in $[x, y]$ for which there exists $x''$ in $[x', y]$ with $|\phi(x'') - \phi(x)| \leq r^{-1} \int_{x'}^y d\lambda_n$. Obviously, $x \in T$. Also $T$ is a closed subset of $[x, y]$ because $|\phi(x') - \phi(x)|$ is a continuous function of $x''$. To show that $T$ is open in $[x, y]$, take any $x'$ in $T$, and choose $x''$ as above. If either $x' = y$ or $x' < x''$, then $[x, x''] \subset T$ is a neighborhood of $x'$ in $[x, y]$. On the other hand, if $x' = x'' < y$, then the above considerations show that there exists $w$ in $(x', y]$ with

$$|\phi(w) - \phi(x')| < r^{-1} |\lambda_n(w) - \lambda_n(x')|.$$ 

Thus we have

$$|\phi(w) - \phi(x)| \leq |\phi(w) - \phi(x')| + |\phi(x') - \phi(x)| < r^{-1} |\lambda_n(w) - \lambda_n(x')| + r^{-1} \int_{x'}^w d\lambda_n \leq r^{-1} \int_x^w d\lambda_n.$$

Therefore $w \in T$ so that $[x, w] \subset T$. Thus $[x, w]$ is a neighborhood of $x'$ in $[x, y]$. Hence $T$ is both open and closed in $[x, y]$. Since $x \in T$, $T = [x, y]$. Therefore $y \in T$, so that $|\phi(y) - \phi(x)| \leq r^{-1} \int_x^y d\lambda_n$ for all $x$ and $y$ in $[t, u)$. Therefore $\phi$ has bounded variation on $[t, u)$, so that $\phi [t, u]$ is a rectifiable sub-arc of $C$. This contradicts the hypothesis. This contradiction shows that $\alpha_{n-1}(t) = 0$ for all $t$ in $[0, 1)$, so that $\alpha_{n-1} = 0$. Having proved this, we can use the same argument to show step by step that $\alpha_{n-2} = \alpha_{n-3} = \cdots = \alpha_0 = 0$. But $\alpha_0(t) = \int_0^t d\mu_0(t)$. Thus $\mu_0$ vanishes on all subsets of $[0, 1)$. Since there is inherent symmetry between the endpoints, $\mu_0 = 0$. Then

$$0 = \alpha_1(t) = \int_0^t [\phi(x) - \phi(t)] d\mu_0(x) + \int_0^t d\mu_1(x) = \int_0^t d\mu_1(x),$$

so that $\mu_1 = 0$. Thus we show step-by-step that $\mu_0 = \mu_1 = \cdots = \mu_n = 0$. This completes the proof of Theorem 1.
There exists Jordan arcs for which condition (2) is fulfilled. For instance, a Jordan arc which has no rectifiable sub-arcs and which has a tangent at a dense set of points will do, because the existence of a tangent implies that a Lipschitz condition of order 1 is fulfilled. To see this, assume that $C$ has a tangent at $\phi(t_0)$. By this we mean that the parameter $t$ can be so chosen that $\phi'(t_0)$ exists and is not zero. Now if $C$ does not satisfy a Lipschitz condition of order 1 at $\phi(t_0)$, then for each $\delta > 0$ there exists $z$ with $|\phi(t_0) - z| < \delta$ such that $d(\phi[0, t_0], z) < |\phi(t_0) - z|/4$ and $d(\phi[t_0, 1], z) < |\phi(t_0) - z|/4$. Therefore, there exist $t_1$ in $[0, t_0]$ and $t_2$ in $[t_0, 1]$ with

$$|\phi(t_1) - z| < |\phi(t_0) - z|/4$$

and $|\phi(t_2) - z| < |\phi(t_0) - z|/4$, so that $|\phi(t_1) - \phi(t_2)| < |\phi(t_0) - z|/2$. Also,

$$|\phi(t_0) - z| \leq |\phi(t_0) - \phi(t_1)| + |t_1 - z| + |\phi(t_0) - \phi(t_1)| + |\phi(t_0) - z|/4,$$

so that $3|\phi(t_0) - z|/4 < |\phi(t_0) - \phi(t_1)|$. Thus,

$$|\phi(t_1) - \phi(t_2)| < 2|\phi(t_0) - \phi(t_1)|/3.$$

It follows that

$$\gamma = |\phi(t_1) - \phi(t_2)| \leq |t_1 - t_2|^{-1} |\phi(t_0) - \phi(t_1)| |t_0 - t_1|^{-1} < 2/3.$$

On the other hand, as $\delta \to 0$ the quantity $\gamma$ converges to $|\phi'(t_0)| \cdot |\phi'(t_0)|^{-1} = 1$. This contradiction shows that $C$ satisfies a Lipschitz condition of order 1 at $\phi(t_0)$. To construct a Jordan arc which has no rectifiable sub-arcs and which has a tangent at a dense set of points, let $f$ be any continuous real function on $[0, 1]$ such that $f'$ exists at a dense set $S$ of points and such that in any sub-interval the set of points where $f'$ does not exist has positive measure. Let $\phi(t) = t + if(t)$, so that $\phi[0, 1] = C$ is a Jordan arc having a tangent at the dense set of points $\phi(S)$. Also $C$ has no rectifiable sub-arcs, because if $\phi[t, u]$ were a rectifiable sub-arc then $\phi$ would be of bounded variation on $[t, u]$, which would imply that $f'$ would exist almost everywhere in $[t, u]$, contrary to the condition on $f$.

It only remains to construct the function $f$. The standard techniques for the construction of nondifferentiable functions can be used (see Hobson [2]). Let $\{r_n\} = s$ be a sequence of irrational numbers which is dense in $[0, 1]$. It is easy to construct inductively a sequence $\{C_n\}$ of countable subsets of $[0, 1]$, each consisting of rational numbers and each containing 0 and 1, such that the accumulation points of $C_n$ are exactly $r_1, \cdots, r_n$, such that the distance
of two consecutive points \( t_1 \) and \( t_2 \) of \( C_n \) is not larger than \( 6^{-n}d \), where 
\( d \) is the distance between the sets \( \{ t_1, t_2 \} \) and \( \{ r_1, \ldots, r_n \} \), and such 
that \( D_n \subseteq C_{n+1} \). Here \( D_n = C_n \cup \mathcal{C}_n \), where \( \mathcal{C}_n \) consists of all points 
which lie midway between two consecutive points of \( C_n \). Thus \( D_n \)
consists of rational numbers, and therefore is disjoint from the se-
queness \( s \). Define the function \( f_n \) on \( [0, 1] \) as follows. Let \( f_n(t) = 0 \) if 
\( t \in C_n \cup \{ r_1, \ldots, r_n \} \). For consecutive points \( t_1 \) and \( t_2 \) in \( C_n \), let 
\( f_n(t) = 3^n(t - t_1) \) if \( t_1 \leq t \leq (t_1 + t_2)/2 \) and \( f_n(t) = 3^n(t_2 - t) \) if 
\( (t_1 + t_2)/2 \leq t \leq t_2 \). Since any point of \( [0, 1] \) which is not in \( C_n \cup \{ r_1, \ldots, r_n \} \) lies 
between two consecutive points of \( C_n \), this defines \( f_n \) uniquely at all 
points of \( [0, 1] \). The function \( f_n \) is clearly continuous, except possibly 
at the points \( r_1, \ldots, r_n \), and \( |f_n'(t)| \) exists and equals \( 3^n \) if \( t \) is not 
in the set \( D_n \cup \{ r_1, \ldots, r_n \} \). Also, if \( u_1 \) and \( u_2 \) are consecutive 
points of \( D_n \), it is clear that \( |f_n(u_2) - f_n(u_1)| \) at \( u_2 - u_1 \) = \( 3^n \). To in-
vestigate the behavior of \( f_n \) at the point \( r_i \), where \( i \leq n \), let \( t \) be any 
point of \( [0, 1] \) \( \setminus \{ r_1, \ldots, r_n \} \). There exist consecutive points \( t_1 \) and 
\( t_2 \) of \( C_n \) with \( t_1 \leq t \leq t_2 \). By the definition of \( f_n \), we have \( |f_n(t)| \leq 3^n(t_2 - t_1) \). 
By the construction of \( C_n \), we have \( t_2 - t_1 \leq 6^{-n} |t - r_i| \). Thus, \( |f_n(t)| = |f_n(t)| \leq 2^{-n} |t - r_i| \) for all \( t \) in \( [0, 1] \). In particular, we 
see that \( f_n'(r_i) = 0 \), so that \( f_n'(t) \) exists if \( t \) is not in \( D_n \), and that \( f_n \) is 
continuous at \( r_i \). Therefore, \( f_n \) is continuous on \( [0, 1] \). We also see 
that \( 0 \leq f_n(t) \leq 2^{-n} \) for all \( t \). The series \( \sum_{n=1}^{\infty} f_n \) therefore converges 
uniformly on \( [0, 1] \) to a continuous function \( f \). It will be shown 
that \( f'(t) \) exists if and only if \( t \) is a member of the sequence \( s \). If 
\( t = r_i \) is a member of \( s \), let \( f = g + h \), where \( g = \sum_{n=1}^{i-1} f_n \) and \( h = \sum_{n=i}^{\infty} f_n \). 
Then \( g'(r_i) \) exists because \( r_i \) is not in \( D_n \) for all \( n \). Also,

\[
| h(u) - h(r_i) | \leq \sum_{n=i}^{\infty} |f_n(u) - f_n(r_i)| \leq \sum_{n=i}^{\infty} 2^{-n} |u - r_i|^2 \leq |u - r_i|^2
\]

for all \( u \), so that \( h'(r_i) \) exists and is zero. Therefore \( f'(r_i) \) exists. Now 
assume that \( t \) is not an element of the sequence \( s \). Let \( n \) be arbitrary. 
Since \( t \) is not an accumulation point of \( C_n \), there exist consecutive 
points \( u_1 \) and \( u_2 \) in \( D_n \) with \( u_1 \leq t \leq u_2 \). Then \( f(u_2) - f(u_1) = \sum_{k=1}^{n} f_k(u_2) - f_k(u_1) \) because \( f_k(u) = 0 \) if \( u \in D_n \subseteq C_{n+1} \) and \( k \geq n + 1 \). Thus

\[
| f(u_2) - f(u_1) | (u_2 - u_1)^{-1} \geq | f_n(u_2) - f_n(u_1) | (u_2 - u_1)^{-1} - \sum_{k=1}^{n-1} | f_k(u_2) - f_k(u_1) | (u_2 - u_1)^{-1} \geq 3^n - \sum_{k=1}^{n} 3^k \geq n.
\]

Thus the difference quotients, \( [f(u_2) - f(u_1)](u_2 - u_1)^{-1} \), for \( u_1 \leq t \leq u_2 \),
are not bounded. It follows that $f'(t)$ does not exist, as was to be proved.

References

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AN IDENTITY IN THE THEORY OF THE GENERALIZED POLYNOMIALS OF JACobi

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1. Introduction of some new notations in the theory of the Jacobi polynomials. To facilitate the passage from the usual Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ to the generalized Jacobi polynomials $P_n^{(\alpha_0, \ldots, \alpha_p)}(x)$ considered here, we introduce some new notations in the theory of the first mentioned polynomials. It is well known\(^1\) that the zeros of these polynomials are the points $x_1 = x_1^{(n)}$, $x_2 = x_2^{(n)}$, \ldots, $x_n = x_n^{(n)}$, which maximize the expression

$$T(x_1, x_2, \ldots, x_n) = T(x) = \prod_{k=1}^n \left(1 - x_k\right)^{p} \left(1 + x_k\right)^{q} \prod_{1 \leq \rho < \omega \leq n} |x_\rho - x_\omega|$$

in the unit-interval $I: [-1, +1]$. Here $\alpha = 2p - 1$ and $\beta = 2q - 1$ and it is assumed $x_1 > x_2 > \cdots > x_n$. Instead of $T(x)$ we use the expression

$$V_m(\xi_1, \xi_2, \ldots, \xi_m; e_1, e_2, \ldots, e_m) = V_m(\xi; e) = \prod_{1 \leq i < k \leq m} (\xi_i - \xi_k)^{e_i e_k},$$

where we suppose that $m = n + 2$; that the points $\xi_1$ and $\xi_m$ are fixed from the outset and are equal to $a_0 = -1$ and $a_1 = +1$ respectively; that $e_1 = p_0 = q$, $e_m = p_1 = p$, $e_2 = e_3 = \cdots = e_{m-1} = 1$; that the points $\xi_1, \xi_2, \ldots, \xi_m$ are counted in increasing order; $-1 = \xi_1 < \xi_2 < \cdots < \xi_{m-1} = +1$ and therefore that $\xi_2 = x_m$, $\xi_3 = x_{m-1}$, \ldots, $\xi_{m-1} = x_1$. It results that $V_m(\xi_1, \xi_2, \ldots, \xi_m; e_1, e_2, \ldots, e_m)$ is a function of $\xi_2, \xi_3, \ldots, \xi_{m-1}; p_0, p_1$ only, as is $T(x)$. Then the zeros of the Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ are the points $\xi_2 = \xi_2^{(m)} = x_m^{(n)}$, $\xi_3 = \xi_3^{(m)} = x_{m-1}^{(n)}$, \ldots, $\xi_{m-1} = \xi_{m-1}^{(m)} = x_1^{(n)}$, which maximize the absolute value of $V_m(\xi; e)$ on $I$, under the mentioned conditions. We call the last function the generalized Vandermondean of the degree $m$ and of the order 1. We write

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