A NOTE ON DERIVATIONS OF LIE ALGEBRAS II

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Let $L$ be a Lie algebra over a ground field of characteristic 0 and let $D(L)$ and $I(L)$ denote the Lie algebra of derivations and inner derivations of $L$ respectively.

The following theorem is proved, although not explicitly stated, in an earlier note.¹

**Theorem 1.** Let $L = S + R$ (Levi decomposition) and let $I(S) = \{D \in D(L), D(S) = 0\}$. Further let $\rho$ denote the restriction homomorphism of $D(L)$ into $D(R)$. Then $D(L)$ splits over $I(L) \implies \rho(I(S))$ splits over $\rho(I(S)) \cap I(R)$.

If $S$ is a semi-simple Lie algebra and $M$ is any $S$-module then $M$ is the direct sum of $M^S$ and $S \cdot M$ where $M^S$ is the trivial submodule of $M$.

**Theorem 2.** Let $L = S + R$ (Levi decomposition). If $R^S \subseteq Z(R)$ (the center of $R$) then $D(L)$ splits over $I(L)$.

**Proof.** Let $u \in R$ and suppose that the derivation of $R$ that is effected by $u$ is the restriction to $R$ of a derivation of $L$ that annihilates $S$. Then $[S, u] \subseteq Z(R)$. As an $S$-module, $R$ is the direct sum of $Z(R)$ and a complementary submodule $P$. The component of $u$ in $P$ is annihilated by $S$ so that $u \in R^S + Z(R) \subseteq Z(R)$. This means

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¹ Details will be found in G. F. Léger, *A note on the derivations of Lie algebras*, Proc. Amer. Math. Soc. vol. 4 (1953). We refer to this note as I.

² The author is grateful to the referee for putting Theorem 2 in this form.
that the derivation of $R$ effected by $u$ is 0. Now it follows from Theorem 1 that $D(L)$ splits over $I(L)$.

At the end of I is the following example (Hochschild’s) of a nilpotent Lie algebra $R$ such that $D(R)$ does not split over $I(R)$. $R$ has a basis $(x_1, x_2, x_3, x_4)$ over a ground field $K$ (of characteristic 0) where $(x_2, x_3, x_4)$ is abelian and $[x_1, x_2] = x_4$, $[x_1, x_3] = 0$, $[x_1, x_4] = 0$.

Now let $L = S + R$ where $S$ has a basis $(s_1, s_2, s_3)$ over $K$ with $[s_i, x_j] = 0$, for $i = 1, 2, 3$ and $j = 3, 4$,

\[
\begin{align*}
[s_1, s_2] &= 2s_2, & [s_2, s_1] &= s_1, & [s_2, x_2] &= x_1, \\
[s_1, s_3] &= -2s_3, & [s_1, x_2] &= -x_2, & [s_3, x_1] &= x_2, \\
[s_1, x_1] &= x_1, & [s_2, x_1] &= 0, & [s_3, x_2] &= 0.
\end{align*}
\]

Then $S$ is simple and $L = S + R$ is a Levi decomposition of $L$. Further $R^S \subseteq Z(R)$ so that $D(L)$ splits over $I(L)$ by Theorem 2. Thus we have an example of a Lie algebra $L$ with radical $R$ such that $D(L)$ splits over $I(L)$ but $D(R)$ does not split over $I(R)$.

This settles a question left open in I.

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