ON A DIVERGENT TRIGONOMETRICAL SERIES GIVEN BY STEINHAUS

SI O B H A N O ' S H E A

Steinhaus [1; 2, p. 283] gave the series

\[ \sum_{n=2}^{\infty} (\log n)^{-1} \cos n(x - \log \log n) \]

as an example of an everywhere-divergent trigonometric series with coefficients tending to zero. Plainly, a sine series cannot diverge everywhere, since it must converge whenever \( x \equiv 0 \pmod{\pi} \). There is, however, no a priori reason why a cosine series should not diverge everywhere. It is not immediately clear from Steinhaus's argument [1] whether the "cosine part" of (1), namely

\[ \sum_{n=2}^{\infty} (\log n)^{-1} \cos (n \log n) \cos nx, \]

has any points of convergence. Accordingly I exhibit here a class of everywhere-divergent cosine series, of which (2) is a special case.

**Theorem.** Suppose that \( u(n) \uparrow \infty \), \( c_n \downarrow 0 \) as \( n \to \infty \), and that there exists a sequence of positive integers \( \{p_n\} \) such that

\[ \limsup_{n \to \infty} (n + p_n) \left\{ u(n + p_n) - u(n) \right\} < \frac{1}{2}, \]

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Then the cosine series
\[
\sum_{n=1}^{\infty} c_n \cos n\mu(n) \cos nx
\]
diverges for all real \(x\), and the sine series
\[
\sum_{n=1}^{\infty} c_n \sin n\mu(n) \sin nx
\]
diverges for all \(x \neq 0 \pmod{\pi}\).

The method used to prove this theorem is a refinement of that employed by Steinhaus to prove the divergence of (1). Put
\[
A_r(x) = \cos r\mu(r) \cos rx - \frac{1}{2} \cos 2rx
\]
and suppose that
\[
u(n) \leq x < \mu(n + 1), \quad n + 1 \leq r \leq n + p_n.
\]
Then
\[
\left| A_r(x) - \frac{1}{2} \right| = \left| \cos rx \{ \cos r\mu(r) - \cos rx \} \right|
\]
\[
= \left| 2 \cos rx \sin \frac{1}{2} r \{ x + \mu(r) \} \sin \frac{1}{2} r \{ x - \mu(r) \} \right|
\]
\[
\leq r \left| x - \mu(r) \right|
\]
\[
\leq (n + p_n) \{ \mu(n + p_n) - \mu(n) \}.
\]
By (3) there exist an integer \(n_0\) and a number \(\lambda < 1/2\) such that this is less than \(\lambda\) for all \(n > n_0\). Thus for all \(n\) and \(x\) satisfying
\[
n > n_0, \quad \mu(n) \leq x < \mu(n + 1),
\]
we have
\[
\sum_{n=1}^{n + p_n} c_r A_r(x) > \left( \frac{1}{2} - \lambda \right) \sum_{n+1}^{n+p_n} c_r.
\]
Every value of \(x\) satisfies (8), modulo \(2\pi\), for an infinity of values of \(n\), and the left-hand side of (9) has period \(2\pi\). Thus, for every \(x\), (9) is true for an infinity of \(n\). Hence by (4),
\[ \limsup_{n \to \infty} \sum_{n+1}^{n+p_n} c_r A_r(x) > 0. \]

But since \( c_n \downarrow 0 \), we have for every \( x \)

\[ \liminf_{n \to \infty} \sum_{n+1}^{n+p_n} c_r \cos 2rx \geq 0, \]

and so by (7),

\[ \limsup_{n \to \infty} \sum_{n+1}^{n+p_n} c_r \cos ru(r) \cos rx > 0. \]

By the general principle of convergence, this proves the divergence of (5).

I omit the proof of the divergence of (6) for all \( x \not\equiv 0 \pmod{\pi} \), which is similar. This proves the theorem.

To deduce from the theorem the divergence of (2) everywhere, we take

\[ u(n) = \log \log n, \quad c_n = (\log n)^{-1}(n \geq 2), \quad p_n = \left\lfloor \frac{1}{4} \log n \right\rfloor, \]

where the square brackets denote the integer part. Then \( u(n) \uparrow \infty \), \( c_n \downarrow 0 \), and the left-hand sides of (3) and (4) are both 1/4. Thus (2) diverges everywhere, by the theorem. Similarly the "sine part" of (1) diverges for all \( x \not\equiv 0 \pmod{\pi} \).

It is not worth while to examine in detail the order of magnitude of those sequences \( \{c_n\} \) for which \( u(n) \) and \( p_n \) can be chosen to satisfy (3) and (4). However, it is easy to show that the choice

\[ c_n = (\log n)^a_1 (\log_2 n)^a_2 (\log_3 n)^a_3 \cdots (\log_q n)^a_q, \]

where \( q \) is a fixed integer and \( \log_m \) denotes the \( m \)th iterated logarithm, is permissible if \( c_n \downarrow 0 \) and \( \sum n^{-1}c_n = \infty \), but not otherwise.

Note added in proof, December 4, 1958. If, in (5), we replace \( \cos nu(n) \) by \( 2 + \cos nu(n) \), we obtain, as may easily be seen, an everywhere-divergent cosine series with non-negative coefficients tending to zero.

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References


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