A GEOMETRIC CONSTRUCTION OF THE $M$-SPACE CONJUGATE TO AN $L$-SPACE$^1$

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1. Introduction. In a previous paper [3] the author described a geometric construction of (abstract) $L$-spaces based upon the characterization of such spaces by R. E. Fullerton [6]. The construction to be introduced here is analogously motivated by the geometric characterization of $C$-spaces given by J. A. Clarkson [4]. Clarkson showed that the unit sphere in a $C$-space is the intersection of a translate of the positive cone with the negative of that translate. Letting $C$ denote the positive cone in an $L$-space $X$ with $F$-unit $e$ we will show that the space of all points lying on rays from the origin $\Theta$ through the set $S=(C-e)\cap(e-C)$, when given an appropriate norm and partial ordering, is an $M$-space with unit element $e$ and unit sphere $S$ and is lattice isomorphic and isometric to the space $L^\infty(\Omega, m)$ conjugate to the concrete representation $L(\Omega, m)$ of $X$. To do this we rely heavily upon the results in the classic papers of S. Kakutani [7, 8].

2. Preliminaries. Let $X$ be a real Banach space with norm $\|x\|$ and additive identity element $\Theta$ which is a linear lattice under the partial ordering relation $x \geq y$ (cf. G. Birkhoff [2]). Let $x \vee y$ and $x \wedge y$ denote the lattice least upper bound and greatest lower bound, respectively, of the elements $x, y \in X$. S. Kakutani [8] calls such a space an (abstract) $M$-space if, in addition, the following conditions are satisfied:

(K.7) $x_n \geq y_n, \|x_n - x\| \rightarrow 0, \|y_n - y\| \rightarrow 0$ imply $x \geq y$.

(K.8) $x \wedge y = \Theta$ implies $\|x + y\| = \|x - y\|$.

(K.9) $x \geq \Theta, y \geq \Theta$ imply $\|x \vee y\| = \max (\|x\|, \|y\|)$.

If condition (K.9) is replaced by

(K.10) $x \geq \Theta, y \geq \Theta$ imply $\|x + y\| = \|x\| + \|y\|$.

Kakutani [7] calls the space an (abstract) $L$-space.

If we let $C = \{x: x \geq \Theta\}$ denote the positive cone in $X$, condition

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(K.7) above may be restated as: "C is closed in the norm topology."^2

An element e of a linear lattice X is called an F-unit if e > 0 and if e ∧ x > 0 for any x > 0. A unit element of a Banach lattice X is an element e ≥ 0 such that ||e|| = 1 and x ≤ e for any x ∈ X with ||x|| ≤ 1. (Cf. [8, p. 997]). It should be recalled from elementary linear space theory that if a convex sub set S of a linear space X is radial at 0, symmetric about 0, and circled then the Minkowski functional p(x) = inf {a: x/a ∈ S, a > 0} is a pseudo-norm on X. (Cf. [5]).

Let L(Ω, m) denote the space of all measurable real valued functions x(t) which are integrable over a compact Hausdorff space Ω with respect to a completely additive measure m such that m(Ω) = 1. A well known result of S. Kakutani [7] tells us that for any L-space X with F-unit there exists such a space L(Ω, m) lattice isomorphic and isometric to X and called the concrete representation of X. We notice that if e is an F-unit of X and a is any positive real number then ae is also an F-unit. Hence we shall assume without loss of generality that the F-unit distinguished in X has unit norm.

Since, in the sequel, certain results we wish to obtain concerning subsets of the L-space X will be more easily demonstrated by using their images in L(Ω, m) and since no confusion nor loss of generality will occur, we will often replace the subset of X by its image in the concrete representation of X without changing notation and without specifically mentioning that the implied equality is in actuality a lattice isomorphism and isometry.

3. The construction.

Theorem. Let X be an L-space with F-unit e, partial ordering x ≥ y, positive cone C, norm ||x|| and unit sphere U = {x: ||x|| ≤ 1, x ∈ X}. Define S = (C − e) ∩ (e − C), p(x) = inf {a: x/a ∈ S, a > 0} and Y = {aS: a ≥ 0}. Then Y is an M-space with unit element e, partial ordering x ≥ y, positive cone C ∩ Y, norm p(x) and unit sphere S = {x: p(x) ≤ 1, x ∈ Y} ⊆ U. Furthermore, Y is lattice isomorphic and isometric to the space L*(Ω, m) conjugate to the concrete representation L(Ω, m) of X.

In order to clarify what would otherwise be a lengthy proof we will demonstrate the theorem by proving the following sequence of lemmas.

^2 It has been communicated to the author by J. E. Kist that axiom (K.7) is redundant since the positive cone C in any real Banach lattice is strongly closed. To see this we need only note that the positive cone is the set C = {x: x ∈ X, x − |x| = Θ} and hence is the inverse image of the closed set {Θ} under the continuous map x → x − |x|.
3.1. **Lemma.** $Y$ is a normed linear space with norm $p(x)$ and unit sphere $S \subset U$.

**Proof.** Using the obvious convexity and symmetry of $S$ it is easy to verify that $Y$ is a real linear space. The fact that $S$ is also circled and radial at $\Theta$ in $Y$ shows that $p(x)$ is a pseudo-norm on $Y$. Let $x$ be any element of $S$. Then, since $S = (C-e) \cap (e-C)$, we have $x = ay - e = e - bz$ where $y$ and $z$ are elements in $C$ of unit norm and $a$ and $b$ are nonnegative real numbers. Now $2x = ay - bz$ and $2e = ay + bz$ so $2\|x\| \leq a\|y\| + b\|z\| = a + b$ and $2 = 2\|e\| = a\|y\| + b\|z\| = a + b$. (Here we use the facts that $X$ is an $L$-space, in particular satisfying (K.10), and that $\|e\| = 1$.) Thus $x \in S$ implies $\|x\| \leq 1$, i.e. $S \subset U$. If $p(x) = 0$ then $p(ax) = 0$ for any real number $a$. It is an easily demonstrated property of $p(x)$ that $\{x: p(x) < 1\} \subset S$. Thus $p(x) = 0$ implies $ax \in S \subset U$ for every real number $a$. This is possible only if $x = \Theta$ hence $p(x)$ is a norm on $Y$. To show that $S = \{x: x \in Y, p(x) \leq 1\}$ we need only show that $p(x) = 1$ implies $x \in S$. Suppose, therefore, that $x \in Y$ with $p(x) = 1$. Define $y_n = (1 - 1/n)x$ for $n = 1, 2, 3, \ldots$. Then $p(y_n) < 1$ and $y_n \in S$ (the closure of $S$ relative to the norm topology of $X$). Both $C - e$ and $e - C$ are closed sets in $X$ and hence $S = \overline{S}$. Thus $x \in S$ and this lemma is proved.

3.2. **Lemma.** $e$ is a unit element of $Y$.

**Proof.** The set $C \cap Y$ is clearly a cone in $Y$. We define a partial ordering in $Y$ by: $x \geq y$ if and only if $x - y \in C \cap Y$ and notice immediately that this is exactly the ordering in $X$ restricted to $Y$. Certainly $e \geq \Theta$ in $Y$ since $e > \Theta$ in $X$. Since $e \in S$, by Lemma 3.1, $p(e) \leq 1$. Suppose that $p(e) < 1$. Then, by definition of $p(x)$, there exists a real number $a$, $0 < a < 1$, such that $1/a \cdot e \in S = (C-e) \cap (e-C)$. But $1/a \cdot e \in e - C$ implies $(a - 1)e \in C$ and since $a - 1 < 0$ this contradicts that $e \in C$. Thus $p(e) = 1$. By definition of $S$, $x \in S$ if and only if $-e \leq x \leq e$ so $p(x) \leq 1$ implies $x \leq e$. This lemma is proved.

3.3. **Lemma.** In $L(\Omega, m)$ we have $S = \{x(t): |x(t)| \leq 1 \text{ a.e.}\}$, $p(x) = \inf \{a: |x(t)| \leq a \text{ a.e., } a > 0\}$, and $Y = \{x(t): \text{there exists an } a > 0 \text{ for which } |x(t)| \leq a \text{ a.e.}\}$.

**Proof.** As mentioned in §2. We make no distinction between sets in $X$ and their equivalent images in $L(\Omega, m)$. Thus, saying that in $L(\Omega, m)$, $S = \{x(t): |x(t)| \leq 1 \text{ a.e.}\}$, we mean that the image of $S$ under the equivalence mapping is exactly the set of all functions $x(t) \in L(\Omega, m)$ which are bounded in absolute value by 1 almost everywhere on $\Omega$. That this is indeed the case follows from the facts that $e$ corresponds to the function $e(t) \equiv 1$ a.e. (cf. [7]), $x \geq \Theta$ in $L(\Omega, m)$
if and only if \( x(t) \geq 0 \) a.e. and that \( x \in S \) if and only if \(-e \leq x \leq e\). The other two equations of the lemma follow easily.

### 3.4. Lemma. \( Y \) is complete under \( p(x) \).

**Proof.** We identify \( Y \) with its image in \( L(\Omega, m) \) and use Lemma 3.3. (Completeness is a property invariant under the equivalence mapping from \( X \) onto \( L(\Omega, m) \).) If \( x \) is any element of \( Y \) then there exists an \( a > 0 \) such that \( |x(t)| < a \) a.e. and hence \( x(t) \) is an essentially bounded function—i.e. \( x(t) \) is in \( L^\infty(\Omega, m) \). Let \( \{x_n\} \) be any Cauchy sequence in \( Y \). That is, given \( e > 0 \) there exists an integer \( N(e) \) such that \( \inf \{a: |x(t) - x_m(t)| < a \) a.e., \( a > 0\} < e \) for all \( n, m \geq N(e) \). In particular, for \( n, m \geq N(e) \), \( |x_n(t) - x_m(t)| < e \) a.e. so \( \{x_n(t)\} \) is a Cauchy sequence in \( L^\infty(\Omega, m) \). Since \( L^\infty(\Omega, m) \) is complete in the norm \( \|x\|_\infty = \text{ess sup } |x(t)| = \inf \{(\sup \{|x(t)| : t \in E\}): m(E) = 0\} \) there exists a function \( y(t) \in L^\infty(\Omega, m) \) such that given any \( e > 0 \) there exists an \( N'(e) \) such that \( \|x_n - y\|_\infty < e \) when \( n \geq N'(e) \). Thus, for \( n \geq N'(e) \) we have \( |x_n(t) - y(t)| < e \) a.e. and, by Lemma 3.3, \( x_n - y \in Y \) so, clearly, \( y \in Y \). Finally, since \( |x_n(t) - y(t)| < e \) a.e. implies \( p(x_n - z) < e \) we have that \( Y \) is complete. This lemma is proved.

### 3.5. Lemma. \( Y \) is an \( M \)-space.

**Proof.** That \( Y \) is a partially ordered linear space under the relation \( x \geq y \) defined by \( x \geq y \) if and only if \( x - y \in \mathcal{C} \cap Y \) is easily seen to be a consequence of the fact that \( \mathcal{C} \cap Y \) is a cone such that \( [\mathcal{C} \cap Y] \cap [-\mathcal{C} \cap Y] = \{0\} \). To show that \( x \vee y \) exists in \( Y \) consider again the concrete representation \( L(\Omega, m) \) of \( X \). In \( L(\Omega, m) \), \( (x \vee y)(t) = \max \{x(t), y(t)\} \). If \( x \) and \( y \) are in \( Y \) then there exists real numbers \( a > 0 \) and \( b > 0 \) such that \( |x(t)| \leq a \) a.e. and \( |y(t)| \leq b \) a.e. Then \( |(x \vee y)(t)| \leq \max (a, b) \) a.e. so \( x \vee y \in Y \). Clearly \( x \wedge y = [-x \vee (-y)] \). Thus \( Y \) is a linear lattice. We shall verify conditions (K.8) and (K.9) to complete the proof of this lemma. To prove (K.8) and (K.9) are satisfied we first use a technique of Clarkson [4].

Assert that for any \( k > 0 \), \( p(x) \leq k \) if and only if \( -ke \leq x \leq ke \). Secondly assert that for any \( x \in Y \), \( -p(x)e \leq x \leq p(x)e \). To prove the first assertion we note that \( p(x) \leq k \) if and only if \( x/k \in S \) or, equivalently, \( -e \leq x/k \leq e \). The second assertion is trivial for \( x = \Theta \) and if \( x \neq \Theta \) it follows from the first by setting \( k = p(x) \). Now assume that \( \Theta \leq x \leq y \) in \( Y \). If \( y = \Theta \) certainly \( p(x) \leq p(y) \). If \( y \neq \Theta \) then \( -p(y)e \leq x \leq y \leq p(y)e \) and, by the first assertion above, \( p(x) \leq p(y) \). We now prove that (K.9) is satisfied. Assume that \( x \geq \Theta, y \geq \Theta \). Since \( x \leq x \vee y \) and \( y \leq x \vee y \) we have \( p(x) \leq p(x \vee y) \) and \( p(y) \leq p(x \vee y) \), and hence \( \max \{p(x), p(y)\} \leq p(x \vee y) \). Without loss of generality assume that
0 ≤ \rho(y) ≤ \rho(x). Since x ≤ \rho(x)e and y ≤ \rho(y)e ≤ \rho(x)e we have x \lor y ≤ \rho(x)e so that \rho(x)e ≤ x \lor y ≤ \rho(x)e and \rho(x \lor y) ≤ \rho(x) = \max [\rho(x), \rho(y)]. Finally, to prove (K.8) let x \land y = \emptyset and again assume that 0 ≤ \rho(y) ≤ \rho(x). Now x + y = x \lor y so \rho(x + y) = \rho(x \lor y) = \max [\rho(x), \rho(y)] = \rho(x). Also 2x = (x + y) + (x - y) so 2\rho(x) ≤ \rho(x + y) + \rho(x - y) = \rho(x) + \rho(x - y). Thus \rho(x) = \rho(x + y) ≤ \rho(x - y).

To get the reverse inequality note that \rho(x + y)e ≥ - (x + y) ≥ x - y ≥ x - y ≥ \rho(x) - \rho(y)e so \rho(x - y) ≥ \rho(x + y). Thus (K.8) holds and this lemma is proved.

3.6. Lemma. Y is lattice isomorphic and isometric to the space \(L^\alpha(\Omega, m)\) conjugate to the concrete representation \(L(\Omega, m)\) of \(X\).

Proof. We saw in Lemma 3.3 that the image in \(L(\Omega, m)\) of \(Y\) under the equivalence mapping between \(X\) and \(L(\Omega, m)\) is exactly the set \(\{x(t) : x(t) \in L(\Omega, m), |x(t)| \leq a \) a.e. for some \(a > 0\}\). But this is precisely the set of essentially bounded \(m\)-measurable functions on \(\Omega\). That the isomorphism between \(Y\) and \(L^\alpha(\Omega, m)\) is order preserving is clear since the ordering in \(L^\alpha(\Omega, m)\) is the restriction of the ordering in \(L(\Omega, m)\). Finally that the mapping is an isometry is seen by noticing that \(\{x(t) : |x(t)| \leq 1 \) a.e.\} coincides with the unit sphere in \(L^\alpha(\Omega, m)\). This lemma and the theorem are proved.

3.7. Corollary. For any two choices \(e_1\) and \(e_2\) of the \(F\)-unit in \(X\) the corresponding \(M\)-spaces \(Y_1\) and \(Y_2\) are isometric. Furthermore, the corresponding measure spaces are in one-to-one measure preserving correspondence modulo sets of measure zero.3

4. Remark. If the \(L\)-space \(X\) contains no \(F\)-unit then \(X\) is a direct sum \(\sum_\alpha \{X_\alpha\}\) of \(L\)-spaces \(X_\alpha\) each with an \(F\)-unit (cf. S. Kakutani [7]). We apply the construction of the theorem to each coordinate space \(X_\alpha\) to arrive at its conjugate space \(X_\alpha^*\). To get \(X^*\), the conjugate of \(X\), we take the direct product \(\prod_\alpha \{X_\alpha^*\}\) of the constructed conjugates.

References


\(^3\) The referee helpfully suggested that this corollary be added.


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**ON** \( u'' + (1 + \lambda g(x))u = 0 \) **FOR** \( \int_0^\infty |g(x)| \, dx < \infty \)

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1. Bellman \([1]\) has raised several questions concerning the solution \( u(x, \lambda) \) of

\[
(1.0) \quad u'' + [1 + \lambda g(x)]u = 0, \quad u(0) = 0, \quad u'(0) = 1
\]

when

\[
(1.1) \quad \int_0^\infty |g(x)| \, dx < \infty.
\]

He states that it is known that for real \( \lambda \)

\[
\lim_{x \to \infty} \left\{ u(x, \lambda) - r(\lambda) \sin [x + \theta(\lambda)] \right\} = 0
\]

where \( r \) and \( \theta \) are functions of \( \lambda \). He asks for the analytic properties of \( r \) and \( \theta \) if \( \lambda \) is a complex variable. In particular he asks whether, if \( g > 0 \), the nearest singularity of \( r \) or \( \theta \) to the origin \( \lambda = 0 \), is on the negative real axis. It will be shown below that it is not. Indeed if \( g \) is real, \( r \) and \( \theta \) are analytic functions of \( \lambda \) for real \( \lambda \).

Let \( g(x) \) be piecewise continuous, (Lebesgue integrable would suffice), and satisfy (1.1). Let

\[
(1.2) \quad B(x) = \int_0^x |g(\xi)| \, d\xi.
\]

**Theorem.** *There is a solution* \( u(x, \lambda) \) *of (1.0) which for each* \( x \) *is an entire function of* \( \lambda \) *and which satisfies*

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