A MAXIMUM MODULUS PROPERTY OF MAXIMAL SUBALGEBRAS

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In a recent paper [6] Wermer considered the algebra $C$ of all continuous complex valued functions on $\gamma$, a simple closed analytic curve bounding a region $\Gamma$, with $\Gamma \cup \gamma$ compact, on a Riemann surface $F$. He considered the subalgebra $A$ of all functions in $C$ which could be extended into $\Gamma$ to be analytic on $\Gamma$ and continuous on $\Gamma \cup \gamma$. Wermer showed that $A$ was a maximal closed subalgebra of $C$ which separated the points of $\gamma$, and that the space of maximal ideals of $A$ was homeomorphic to $\Gamma \cup \gamma$.

In [2] Civin and Yood considered a class of subalgebras of complex commutative regular Banach algebras which become maximal closed subalgebras in the event the original algebra was the collection of continuous functions on a compact Hausdorff space. The object of this note is to demonstrate that such subalgebras possess a maximum modulus property possessed by $A$. To state the result obtained we recall certain definitions. The terms not herein defined may be found in [5].

Let $B$ be a complex commutative regular Banach algebra with identity $e$ and space of maximal ideals $\mathfrak{M}(B)$. Let $\pi: x \mapsto x(M)$ be the Gelfand representation of $B$ as a subalgebra of $C(\mathfrak{M}(B))$, the continuous function on $\mathfrak{M}(B)$. We also denote $\pi x$ by $\hat{x}$ and $\pi Q$ by $\hat{Q}$ for any subset $Q$ of $B$. A subalgebra $N$ of $B$ is called determining [2] if $\pi N$ is dense in $\pi B$, otherwise $N$ is called nondetermining. A subalgebra of $B$ is called a maximal nondetermining subalgebra if every larger subalgebra of $B$ is determining. A subset $S$ of $B$ is called a separating family on $\mathfrak{M}(B)$ if for each $M_1$, $M_2$ in $\mathfrak{M}(B)$, $M_1 \neq M_2$, there exists an $x \in S$ such that $x(M_1) \neq x(M_2)$. If $P$ is an algebra of continuous complex valued functions vanishing at infinity on the locally compact space $X$, the smallest closed set (if it exists) on which each $|f|$ with $f \in P$ assumes its maximum is called the Šilov boundary of $X$ with respect to $P$.

**Theorem 1.** Let $B$ be a complex commutative regular Banach algebra with identity $e$, and let $N$ be a maximal nondetermining subalgebra of

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which is not a maximal ideal. If \( N \) is a separating family on \( \mathcal{M}(B) \), then \( \mathcal{M}(B) \) may be topologically embedded if \( \mathcal{M}(N) \) and as so embedded \( \mathcal{M}(B) \) is the Silov boundary of \( \mathcal{M}(N) \) with respect to \( N \).

While the present note was in the process of publication, two proofs of Theorem 1 appeared for the special case when \( B = C(X) \) for a compact Hausdorff space \( X \), one by H. S. Bear [1] and the other by K. Hoffman and I. M. Singer [4].

Before proceeding to the proof of the theorem, we require one lemma, which was noted by Helson and Quigley [3] for the case \( B = C(X) \).

**Lemma 2.** Let \( N \) be a maximal nondetermining subalgebra of the complex commutative regular Banach algebra \( B \), and let \( e \) be the identity of \( B \). Then either \( e \in N \) or \( N \) is a maximal ideal of \( B \).

Suppose \( e \in N \). Let \( D = \{ a + \lambda e : a \in N \text{ and } \lambda \text{ complex} \} \). As \( e \) is the unit for \( B \), \( D \) is a subalgebra of \( B \) which properly contains \( N \), hence \( \hat{D} \) is dense in \( \hat{B} \). Let \( x \in \mathcal{B} \) and \( a \in N \). There exists \( a_n \in N \) and \( \lambda_n \) complex, \( n = 1, 2, \ldots \), such that \( \pi(a_n + \lambda_n e) \to px \) as in \( n \to \infty \). Therefore \( (a_n + \lambda_n e)a \in N \) and \( \pi \{ (a_n + \lambda_n e)a \} \to \pi(xa) \). By Lemma 1 of [2], \( \hat{N} \) is closed in \( \hat{B} \), so \( \pi(xa) \in \hat{N} \). There thus exists \( u \in N \) such that \( xa - u \) is in the radical of \( B \). As noted in [2], \( N \) contains the radical of \( B \). Thus \( xa \in N \) and \( N \) is an ideal of \( B \). That \( N \) is a maximal ideal is an immediate consequence of \( N \) being maximal nondetermining.

We return to the proof of Theorem 1. Each nonzero multiplicative linear functional on \( B \) is automatically one on \( \mathcal{M}(B) \), and distinct multiplicative linear functionals on \( B \) have distinct restrictions to \( N \) since \( N \) is a separating family on \( \mathcal{M}(B) \). There is thus a one-to-one correspondence between \( \mathcal{M}(B) \) and a subset of \( \mathcal{M}(N) \). The mapping is clearly continuous from \( \mathcal{M}(B) \) to \( \mathcal{M}(N) \) in the Gelfand topologies. As \( \mathcal{M}(B) \) is a compact Hausdorff space, the mapping is a homeomorphism. We henceforth suppose \( \mathcal{M}(B) \) is a subset of \( \mathcal{M}(N) \).

Since \( N \) is a subalgebra of \( B \), \( \lim ||a^n||^{1/n} \) is independent of whether the \( N \) or \( B \) norm is used. Thus \( \sup | a(M) | \) is the same whether calculated over \( \mathcal{M}(B) \) or \( \mathcal{M}(N) \). To see that \( \mathcal{M}(B) \) is the \( \tilde{\text{S}} \)ilov boundary of \( \mathcal{M}(N) \) with respect to \( N \), it is sufficient to see that there is no proper closed subset of \( \mathcal{M}(B) \) on which each \( | a(M) | \), \( a \in N \), attains its maximum. Suppose otherwise and let \( \mathcal{G} \) be a proper closed subset of \( \mathcal{M}(B) \) of the required type.

Let \( M_0 \in \mathcal{M}(B) \), \( M_0 \notin \mathcal{G} \). If \( \mathcal{G} \) is any closed set in \( \mathcal{M}(B) \) such that \( \mathcal{L} \supset \mathcal{G} \) and \( M_0 \notin \mathcal{L} \), let \( \mathcal{B} \) be an open set in \( \mathcal{M}(B) \) with \( M_0 \in \mathcal{B} \) and \( \mathcal{B} \cap \mathcal{L} = \emptyset \), the closure being in \( \mathcal{M}(B) \). Let \( W = W(\mathcal{G}) \) be the kernel of \( \mathcal{G} \), i.e. \( W = \cap M, M \in \mathcal{G} \). Let \( R \) be the radical of \( B \). Since \( B \) is a regular...
Banach algebra, $W$ contains elements not in $R$. Consider the algebra $S = N + W$. The elements of $S$ are of the form $a + u, a \in N, u \in W$, since $W$ is an ideal of $B$. For $u \in W, u \in R$, the maximum modulus of $u(M)$ is not attained on $\mathcal{F}$, so $u \in N$, and thus $S$ contains $N$ properly. As $N$ was maximal nondetermining, $\mathcal{S}$ is dense in $\mathcal{B}$.

Let $b \in B$. There exists $a_n \in N, u_n \in W, n = 1, 2, \ldots$, so that if $r_n = a_n + u_n$, then $\tau_n \to b$. For $M \in \mathcal{S}$, $|a_n(M) - a_m(M)| = |r_n(M) - r_m(M)|$. Thus

$$\sup_{M \in \mathcal{S}} |a_n(M) - a_m(M)| \leq \sup_{M \in \mathcal{B}} |r_n(M) - r_m(M)| .$$

By the maximum modulus property of $N$ relative to $\mathcal{K} \subseteq \mathcal{S}$,

$$\sup_{M \in \mathcal{B}} |a_n(M) - a_m(M)| \leq \sup_{M \in \mathcal{B}} |r_n(M) - r_m(M)| .$$

Since $\mathcal{N}$ is closed [2], there exists $a_0 \in N$ such that $\delta_n \to \delta_0$. There is then an element $w_0 \in W$ such that $a_n \to w_0$. If $b_0 = a_0 + w_0$, $\tau_n \to b$ and $r_n \to b_0$, and consequently $b - b_0 = 0$ and $b - b_0 \in R$. As noted in [2], $R \subseteq N$, so $b - b_0 \in N$. Since $b$ was arbitrary, $B = N + W = N + W(\mathcal{S})$.

We next show the complement of $\mathcal{F}$ in $\mathcal{M}(B)$ consists of a single point. Suppose otherwise. Let $M_i \in \mathcal{M}(B), M_i \in \mathcal{F}, i = 1, 2$, and $M_1 \neq M_2$. Let $\mathcal{G}$ be a closed set in $\mathcal{M}(B)$, such that $\{ M_i \} \cup \mathcal{K} \subseteq \mathcal{G}$ and $M_2 \in \mathcal{G}$. Since $B$ is a regular Banach algebra, there is an element $b \in B$, such that $b(M) = 0$, $M \in \mathcal{K}$, and $b(M_1) = 1$. We may express $b$ as $b = a + u, a \in N, u \in W(\mathcal{K})$. For $M \in \mathcal{K}$, $0 = b(M) = a(M) + u(M)$. Since $u(M) = 0$ for $M \in \mathcal{K}, a(M) = 0$ for $M \in \mathcal{K}$. However, $1 = b(M_1) = a(M_1) + u(M_1) = a(M_1)$ since $M_1 \in \mathcal{K}$. This contradicts the supposition that for $a \in N$,

$$\sup_{M \in \mathcal{K}} |a(M)| = \sup_{M \in \mathcal{B}} |a(M)| .$$

Thus $\mathcal{M}(B) = \mathcal{F} \cup \{ M_0 \}$, and since $\mathcal{F}$ was closed in $\mathcal{M}(B)$, $M_0$ is an isolated point of $\mathcal{M}(B)$.

Let $W = W(\mathcal{F}) = \{ f \in B | f(\mathcal{F}) = 0 \}$. Consider any element $b + W$ of $B/W$. Since $b = a + u$, with $a \in N, u \in W$, there is an element $a$ of $N$ in the coset $b + W$. Now $R \subseteq W$, so all elements of the coset $a + R$ of $N/R$ are in the coset $b + W$. Moreover if $a_i \in b + W$, and $a_i \in N$, $i = 1, 2$, then $a_1 - a_2 \in W$ so by the maximum modulus property that $\mathcal{F}$ is alleged to have $\delta_1 - \delta_2 = 0$ and $a_1 - a_2 \in R$. There is thus a one-to-one correspondence between the cosets $b + W$ and $a + R$. The correspondence gives an isomorphism of $B/W$ and $N/R$.

Let $N_1 = \{ a \in N : a(M_0) = 0 \}$. Then $N_1$ is a maximal ideal of $N$ which contains $R$ and thus $N_1/R$ is a maximal ideal of $N/R$. The iso-
morphism obtained above implies the existence of a maximal ideal $M_1$ in $B$, $M_1 \supset W$ and with $M_1/W$ isomorphic to $N_1/R$. Since $M_1 \supset W$, $M_1 \neq M_0$.

Let $a \in N_1$, so $a(M_0) = 0$. Then $a(M_1) = 0$ because of the inclusion of the coset $a + R$ in the coset $a + W$. Similarly, if $a \in N \cap M_1$, then $a \in M_0$. Lemma 2 implies that for arbitrary $a \in N$, there is a constant $\lambda$ such that $a - \lambda e \in N_1$. But then $a(M_0) - \lambda = a(M_1) - \lambda$ and $N$ does not separate the points of $\mathfrak{M}(B)$. This contradiction completes the proof of the theorem.

Bibliography


