A REMARK ON CONTINUITY CONDITIONS

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In a recent paper [1, p. 207], as a corollary to a theorem on Fourier series with gaps, I pointed out that a function of one real variable may satisfy a Lipschitz condition in a set of positive measure without satisfying such a condition in any interval. Here a Lipschitz condition in a general set of real numbers is defined as follows: \( f(x) \in \text{Lip } \alpha \) in \( E \) if

\[
f(x + h) - f(x) = O(\mid h \mid ^{\alpha})
\]

uniformly for \( x \) in \( E \), as \( h \to 0 \) through unrestricted real values. When writing [1] I overlooked the fact that it is possible by simpler methods to obtain a much stronger result. Let \( E \) be a subset of \((0, 1)\), \( f(x) \) a function defined in \((0, 1)\) and \( \omega(t) \) a function defined, positive and monotonic increasing in \((0, 1)\) and satisfying \( \omega(t) \to 0 \) as \( t \to 0 \). Let us then call \( \omega(t) \) a modulus of continuity of \( f(x) \) in \( E \) if

\[
|f(x + h) - f(x)| \leq \omega(\mid h \mid )
\]

for all \( x \) in \( E \) and all \( x + h \) in \((0, 1)\) \((h \neq 0)\). With this terminology we have the following theorem.

**Theorem.** Let \( E \) be any subset whatever of \((0, 1)\). Let \( \omega(t) \) be positive and monotonic increasing in \((0, 1)\) and satisfy \( \omega(t) \to 0 \) as \( t \to 0 \). Then there exists a function \( f(x) \) defined in \((0, 1)\) such that

(i) \( \omega(t) \) is a modulus of continuity of \( f(x) \) in \( E \),

(ii) \( f(x) \) is discontinuous at every interior point of the complement of \( E \).

Here and in the rest of this note, "complement" means "complement with respect to \((0, 1)\)." In the theorem we may, for instance, choose for \( E \) a nondense closed set of positive measure, whose complement is then an open set dense in \((0, 1)\). Thus we see from the theorem that \( f(x) \) may satisfy as strong a continuity condition as we wish in \( E \), without even being continuous in any interval.

To prove the theorem, denote by \( S \) the interior of the complement of \( E \). We assume that \( S \) is nonempty, since if this is not the case the function \( f(x) \equiv 0 \) has all the required properties. Then \( S \) is the union of countably many disjoint open intervals, say

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Let $T$ denote the complement of $S$, so that $E \subseteq T$. Put $f(x) = 0$ in $T$, and also put $f(x) = 0$ when $x$ is a rational point of $S$. If $x$ is an irrational point of $S$, then $|x - x_n| < \delta_n$ for exactly one value of $n$; in this case put

$$f(x) = \omega(\delta_n - |x - x_n|).$$

Then $f(x)$ is defined throughout $(0, 1)$; we now prove (i) and (ii).

Suppose that $x \in E$ and $x + h \in (0, 1)$ ($h \neq 0$). If either $x + h \in T$ or $x + h$ is a rational point of $S$, then (1) is trivial. Otherwise $x + h$ is an irrational point of $S$, and so of some interval $(x_n - \delta_n, x_n + \delta_n)$, and

$$f(x + h) - f(x) = \omega(\delta_n - |x + h - x_n|).$$

But obviously $|h|$ is not less than the distance of $x + h$ from the nearer of the two points $x_n - \delta_n, x_n + \delta_n$; and this distance is $\delta_n - |x + h - x_n|$. Hence (1) is true, by (2) and the monotonicity of $\omega(t)$. This proves (i).

(ii) is plainly true, from the definitions of $\omega(t)$ and $f(x)$. This proves the theorem.

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Reference


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