A COMPARISON THEOREM FOR ELLIPTIC EQUATIONS\textsuperscript{1}

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Hartman and Wintner \cite{1} obtained a Sturmian comparison theorem for self-adjoint second order linear elliptic equations of the form

\begin{equation}
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + f u = 0, \quad a_{ij} = a_{ji}
\end{equation}

in a bounded domain $D$ with boundary $\Gamma$. It is the purpose of this note to extend their result to general second order linear elliptic equations. Let $v$ be a solution of another elliptic equation of the same form

\begin{equation}
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \alpha_{ij} \frac{\partial v}{\partial x_j} \right) + F v = 0, \quad \alpha_{ij} = \alpha_{ji}
\end{equation}

and denote the elements of the inverse matrices by $a^{ij}$ and $\alpha^{ij}$ respectively.

Let $u(x) = u(x_1, \cdots, x_n)$ be a solution of (1) which is nonnegative in $D$ and vanishes on $\Gamma$. We exclude the trivial case $u \equiv 0$. Suppose that the coefficients of (1) and (2) satisfy the relations

\begin{equation}
||\alpha^{ij} - a^{ij}|| \text{ is non-negative definite in } D,
\end{equation}

\begin{equation}
f \leq F \text{ in } D.
\end{equation}

The theorem of Hartman and Wintner states that if (3) and (4) hold then any solution of (2) must have a zero in $D + \Gamma$.

We now consider in $D$ two general linear elliptic equations of the form

\begin{equation}
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + f u = 0, \quad a_{ij} = a_{ji},
\end{equation}

\begin{equation}
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \alpha_{ij} \frac{\partial v}{\partial x_j} \right) + \sum_{i=1}^{n} \beta_i \frac{\partial v}{\partial x_i} + F v = 0, \quad \alpha_{ij} = \alpha_{ji},
\end{equation}

and determine conditions on the coefficients so that a Sturmian comparison theorem shall be valid. For simplicity we consider the case of two independent variables, the result in more than two vari-

\textsuperscript{1} This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 49(638)-398.

Received by the editors July 26, 1958.
ables being a straightforward extension. That is, we consider the equations

\begin{align*}
L_1u & \equiv (au_x)_x + (bu_y)_y + (cu_y)_x + (fu)_y \\
& \quad + 2du_x + 2eu_y + fu = 0,
L_2v & \equiv (av_x)_x + (bv_y)_y + (cv_y)_x + (fv_y)_y \\
& \quad + 2dv_x + 2ev_y + Fv = 0.
\end{align*}

An application of Green's theorem to (7) for a function \( u \) which vanishes on \( \Gamma \) yields

\[- \int \int_D uL_1u dx dy = \int \int_D \left[ au_x^2 + 2bu_xu_y + cu_y^2 - 2duu_x - 2euu_y - fu^2 \right] dx dy.\]

We now employ a device of Picard [3] extended by Hartman and Wintner [1]. The same technique was also used by Ou and Ding [2] and the author in connection with certain problems in equations of mixed type. Let \( P \) and \( Q \) be arbitrary smooth functions in \( D \). Then for any function \( u \) vanishing on \( \Gamma \) we have

\[ \int \int_D \left[ \frac{\partial}{\partial x} (Pu^2) + \frac{\partial}{\partial y} (Qu^2) \right] dx dy = 0. \]

Thus if in addition \( u \) is a solution of (7) we find

\[ \int \int_D \left[ au_x^2 + 2bu_xu_y + cu_y^2 + 2(P - d)uu_x \\
+ 2(Q - e)uu_y + (P_x + Q_y - f)u^2 \right] dx dy = 0. \]

The condition that the integrand be a positive semi-definite form in \( u, u_x, u_y \) is

\[ (ac - b^2)(P_x + Q_y - f) \geq a(Q - e)^2 - 2b(P - d)(Q - e) + c(P - d)^2. \]

Define

\[ P = -\frac{av_x - \beta v_y}{v} + d - \xi, \quad Q = -\frac{\beta v_x - \gamma v_y}{v} + e - \eta \]

where \( v \) is a positive solution of (8) and \( \xi, \eta \) are functions yet to be determined. Substitution for \( P \) and \( Q \) in (10) yields the expression
\[(ac - b^2)(\alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2) + (ac - b^2)v[2\delta v_x + 2\epsilon v_y + (F - f - \xi_x - \eta_y + d_x + e_y)v] \geq (a\beta^2 - 2b\alpha\beta + c\alpha^2)v_x^2 + 2(a\beta\gamma - b\beta^2 - b\alpha\gamma + c\alpha\beta)v_x v_y + (a\gamma^2 - 2b\beta\gamma + c\beta^2)v_y^2 + 2(a\eta\gamma + c\beta\xi - b\beta\eta - b\gamma\xi)v_x v_y + (a\eta^2 - 2b\xi\eta + c\xi^2)v^2.\]

We now investigate conditions on the coefficients of (7) and (8) which will make the above inequality hold throughout $D$. This will contradict (9) and hence $v$ must have a zero in $D$. If condition (3) is satisfied, which in this case becomes

\[(a - \alpha)(c - \gamma) - (b - \beta)^2 \geq 0,\]

then (11) will be a consequence of the inequality

\[(ac - b^2)v[2\delta v_x + 2\epsilon v_y + (F - f - \xi_x - \eta_y + d_x + e_y)v] \geq 2(a\beta\eta + c\alpha\xi - b\alpha\eta - b\beta\xi)v_x^2 + 2(a\eta\gamma + c\beta\xi - b\beta\eta - b\gamma\xi)v_x v_y + (a\eta^2 - 2b\xi\eta + c\xi^2)v^2.\]

We select $\xi, \eta$ to be solutions of the linear equations

\[
(a\beta - b\alpha)\eta + (c\alpha - b\beta)\xi = \delta(ac - b^2),
(a\gamma - b\beta)\eta + (c\beta - b\gamma)\xi = \epsilon(ac - b^2).
\]

That this is always possible follows from the ellipticity of the operators $L_1$ and $L_2$. Hence (11) is now a consequence of

\[(ac - b^2)(F - f - \xi_x - \eta_y + d_x + e_y) - (a\eta^2 - 2b\xi\eta + c\xi^2) \geq 0.\]

We therefore have the result: if the coefficients of (7) and (8) satisfy relations (12) and (13) and if there is a non-negative solution of (7), not identically zero, which vanishes on $\Gamma$, then every solution of (8) must have a zero in $D + \Gamma$.

In the special case $\delta = \epsilon = 0$ we have $\xi = \eta = 0$ and condition (13) becomes

\[F - f + d_x + e_y \geq 0.\]

This is particularly useful if a comparison is desired between a general linear second order equation and the equation

\[\Delta u + mu = 0\]

$m$ a constant. Then we have the conditions
Of course a more general relation between the coefficients, sufficient to yield the conclusion of the theorem, can be obtained by imposing the condition that (11) be a positive semi-definite form in $v, v_z, v_y$.

BIBLIOGRAPHY

