APPROXIMATION ON A LINE SEGMENT BY BOUNDED ANALYTIC FUNCTIONS: PROBLEM β

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Problem β is the study of degree of approximation on a closed bounded point set E to a function $f(z)$ analytic on E, by functions analytic and bounded in a region D containing E; here $f(z)$ is supposed not analytic throughout D, but to possess certain continuity properties on the boundary of a suitable region of analyticity, which contains E and whose closure lies in D. This is a problem on which considerable progress has recently been made [1] if E is bounded by analytic Jordan curves. It is our present purpose to indicate further progress, now for E a line segment. We prove the

THEOREM. *Let D be a region of the z-plane bounded by a finite number of mutually disjoint Jordan curves C, and let E: $-1 \leqq z \leqq 1$ lie in D. Let the function $u(z)$ be harmonic in $D-E$, continuous on the closure of $D-E$, equal to zero on E and unity on C. Let C_σ denote generically the locus $u(z)=\sigma$, $0<\sigma<1$, in D, and let D_σ denote generically the region $0 \leqq u(z) < \sigma$ in D.*

Let $f(z)$ be analytic throughout D_ρ, of class $L(p, \alpha)$ on C_ρ, $0<\alpha<1$, and suppose C_ρ bounded and without multiple points. Then for every $\lambda(\geqq 1)$ there exist functions $f_\lambda(z)$ analytic in D and satisfying the inequalities

$$(1) \qquad |f(z) - f_\lambda(z)| \leqq Ae^{-\lambda\rho}/\lambda^{p+\alpha}, \qquad z \text{ on } E,$$

$$(2) \qquad |f_\lambda(z)| \leqq Ae^{\lambda(1-\rho)}/\lambda^{p+\alpha}, \qquad z \text{ in } D.$$

Reciprocally, if $f(z)$ is defined on E, if C_ρ is bounded and consists of a finite number of mutually disjoint Jordan curves, and if the $f_\lambda(z)$ exist for every $\lambda \geqq 1$ analytic in D and satisfying (1) *and* (2), *then $f(z)$ can be defined so as to be analytic throughout D_ρ, of class $L(p-1, \alpha)$, $0<\alpha<1$, on C_ρ.*

Here and in the sequel the numbers A with or without subscripts represent constants independent of λ and z. The class $L(p, \alpha)$ on C_ρ for integral p $(\geqq 0)$ requires that $f(z)$ be analytic in D_ρ, continuous in the two-dimensional sense on C_ρ, and that $f^{(p)}(z)$ exist on C_ρ and satisfy there a Lipschitz condition of order α. The class $L(p, \alpha)$ for integral p (<0) on C_ρ requires that $f(z)$ be analytic in D_ρ with

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$|f(z)| \leqq A(\rho-\sigma)^{p+\alpha}$ for z on C_σ, $\sigma<\rho$, where A is constant independent of z and σ. These classes remain invariant under conformal transformation if the point at infinity is not involved.

Let $f(z)$ be given of class $L(p, \alpha)$ on C_ρ. We shall use the classical map, the inverse of $z=(w+1/w)/2$ which transforms E into E_w: $|w|=1$, and D into two regions D' and D'' of the w-plane respectively exterior and interior to E_w. The transform $F(w)$ of $f(z)$ is analytic throughout the two regions D_ρ' and D_ρ'', images of D_ρ, respectively exterior and interior to E_w, and $F(w)$ is of class $L(p, \alpha)$ on the images of C_ρ, parts of the boundaries of D_ρ' and D_ρ''. Thus $F(w)$ is continuous and hence analytic also on E_w. In the region $D_\rho'+D_\rho''+E_w$ the function $F(w)$ is the sum of two components $F_1(w)$ and $F_2(w)$ defined by Cauchy's integral extended over sets of Jordan curves in D_ρ' and D_ρ'' near the boundaries other than E_w of those respective regions; if D_ρ' is infinite we add $F(\infty)$ to the second of these integrals. The components $F_1(w)$ and $F_2(w)$ are analytic throughout D_ρ' plus $|w| \leqq 1$ and D_ρ'' plus $|w| \geqq 1$ respectively, of class $L(p, \alpha)$ on the boundaries.

By the general theory [1, Theorem 6 and §8] there exist families of functions $F_{\lambda 1}(w)$ and $F_{\lambda 2}(w)$ analytic throughout D' plus $|w| \leqq 1$ and D'' plus $|w| \geqq 1$ respectively, hence in particular analytic throughout $D_w=D'+D''+E_w$, satisfying

$$\begin{aligned}
|F_1(w) - F_{\lambda 1}(w)| &\leqq A_1 e^{-\lambda}/\lambda^{p+\alpha}, && w \text{ on } E_w,\\
|F_{\lambda 1}(w)| &\leqq A_2 e^{\lambda(1-\rho)}/\lambda^{p+\alpha}, && w \text{ in } D_w,\\
|F_2(w) - F_{\lambda 2}(w)| &\leqq A_1 e^{-\lambda\rho}/\lambda^{p+\alpha}, && w \text{ on } E_w,\\
|F_{\lambda 2}(w)| &\leqq A_2 e^{\lambda(1-\rho)}/\lambda^{p+\alpha}, && w \text{ in } D_w.
\end{aligned}$$

If we set $F(w) \equiv F_1(w)+F_2(w)$, $F_\lambda(w) \equiv F_{\lambda 1}(w)+F_{\lambda 2}(w)$, we have consequently

$$\begin{aligned}
|F(w) - F_\lambda(w)| &\leqq A_3 e^{-\lambda\rho}/\lambda^{p+\alpha}, && w \text{ on } E_w,\\
|F_\lambda(w)| &\leqq A_4 e^{\lambda(1-\rho)}/\lambda^{p+\alpha}, && w \text{ in } D_w.
\end{aligned}$$

However, D_w is invariant under the transformation $w'=1/w$, as also is $F(w)$, so we have

$$\begin{aligned}
|F(w) - F_\lambda(1/w)| &\leqq A_3 e^{-\lambda\rho}/\lambda^{p+\alpha}, && w \text{ on } E_w,\\
|F_\lambda(1/w)| &\leqq A_4 e^{\lambda(1-\rho)}/\lambda^{p+\alpha}, && w \text{ in } D_w,
\end{aligned}$$

$$|2F(w) - [F_\lambda(w) + F_\lambda(1/w)]|/2 \leqq A_3 e^{-\lambda\rho}/\lambda^{p+\alpha}, \quad w \text{ on } E_w, \tag{3}$$

$$|F_\lambda(w) + F_\lambda(1/w)]|/2 \leqq A_4 e^{\lambda(1-\rho)}/\lambda^{p+\alpha}, \quad w \text{ in } D_w. \tag{4}$$

Under the transformation $z=(w+1/w)/2$, the function

$$[F_\lambda(w) + F_\lambda(1/w)]/2$$

is carried into a function $f_\lambda(z)$ single valued and analytic throughout D; the function $f_\lambda(z)$ is analytic even on E, since it is analytic in a deleted neighborhood of E and continuous in the closure of that neighborhood. Thus (1) and (2) follow from (3) and (4).

In the converse direction, inequalities (1) and (2) imply that $f(z)$ can be extended from E so as to be analytic throughout D_ρ, of class $L(p-1, \alpha)$ on C_ρ; indeed this fact can be proved by reference to the w-plane and Theorem 1 of [1], or can be proved by the method of proof of that theorm directly in the given z-plane, by applying the two-constant theorem to compute degree of convergence on C_ρ $(p \geqq 0)$ or convergence on C_σ near C_ρ $(p<0)$.

If E is an arbitrary circular arc (not an entire circumference) and D is an arbitrary region containing E and bounded by a finite number of mutually disjoint Jordan curves, the theorem just established applies after a suitable linear transformation of the plane is made, and hence applies in appropriate form before this linear transformation is made. Similarly the conclusion applies to an arbitrary analytic Jordan arc E and region D containing E and bounded by a finite number of mutually disjoint Jordan curves, provided there exists a schlicht one-to-one map of the closure of D which carries E into a line segment. But the corresponding theorem where E is an arbitrary analytic Jordan arc and D is arbitrary has not yet been established.

In the theorem we have required the boundary of D to consist of a finite number of mutually disjoint Jordan curves, but this requirement may be weakened. It is sufficient if D is bounded by a finite number of mutually disjoint continua, none of which is a single point, because that is sufficient for application to D_w of the previous theory [1]; a succession of conformal maps, say of such a D_w, transforms D_w into a region bounded by a finite number of mutually disjoint analytic Jordan curves, and transforms E_w into an analytic Jordan curve.

A previous study exists [2, §8.2] of Problem β for approximation by polynomials, which essentially includes the special case of the present theorem in which D is bounded by an ellipse with foci $+1$ and -1. Such approximation by polynomials was first considered by S. Bernstein.

References

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