APPROXIMATION ON A LINE SEGMENT BY BOUNDED ANALYTIC FUNCTIONS: PROBLEM $\beta$

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Problem $\beta$ is the study of degree of approximation on a closed bounded point set $E$ to a function $f(z)$ analytic on $E$, by functions analytic and bounded in a region $D$ containing $E$; here $f(z)$ is supposed not analytic throughout $D$, but to possess certain continuity properties on the boundary of a suitable region of analyticity, which contains $E$ and whose closure lies in $D$. This is a problem on which considerable progress has recently been made [1] if $E$ is bounded by analytic Jordan curves. It is our present purpose to indicate further progress, now for $E$ a line segment. We prove the

**Theorem.** Let $D$ be a region of the $z$-plane bounded by a finite number of mutually disjoint Jordan curves $C$, and let $E: -1 \leq z \leq 1$ lie in $D$. Let the function $u(z)$ be harmonic in $D - E$, continuous on the closure of $D - E$, equal to zero on $E$ and unity on $C$. Let $C_\sigma$ denote generically the locus $u(z) = \sigma$, $0 < \sigma < 1$, in $D$, and let $D_\sigma$ denote generically the region $0 \leq u(z) < \sigma$ in $D$.

Let $f(z)$ be analytic throughout $D_\rho$, of class $L(\rho, \alpha)$ on $C_\rho$, $0 < \alpha < 1$, and suppose $C_\rho$ bounded and without multiple points. Then for every $\lambda (\geq 1)$ there exist functions $f_\lambda(z)$ analytic in $D$ and satisfying the inequalities

\[
\begin{align*}
(1) & \quad |f(z) - f_\lambda(z)| \leq A e^{-\lambda \rho / \lambda^{\rho + \alpha}}, & z & \text{on } E, \\
(2) & \quad |f_\lambda(z)| \leq A e^{\lambda(1-\rho) / \lambda^{\rho + \alpha}}, & z & \text{in } D.
\end{align*}
\]

Reciprocally, if $f(z)$ is defined on $E$, if $C_\rho$ is bounded and consists of a finite number of mutually disjoint Jordan curves, and if the $f_\lambda(z)$ exist for every $\lambda \geq 1$ analytic in $D$ and satisfying (1) and (2), then $f(z)$ can be defined so as to be analytic throughout $D_\rho$, of class $L(\rho - 1, \alpha)$, $0 < \alpha < 1$, on $C_\rho$.

Here and in the sequel the numbers $A$ with or without subscripts represent constants independent of $\lambda$ and $z$. The class $L(\rho, \alpha)$ on $C_\rho$ for integral $\rho (\geq 0)$ requires that $f(z)$ be analytic in $D_\rho$, continuous in the two-dimensional sense on $C_\rho$, and that $f^{(\rho)}(z)$ exist on $C_\rho$ and satisfy there a Lipschitz condition of order $\alpha$. The class $L(\rho, \alpha)$ for integral $\rho (<0$) on $C_\rho$ requires that $f(z)$ be analytic in $D_\rho$ with

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\[ |f(z)| \leq A (\rho - \sigma)^{p+\alpha} \] for \( z \) on \( C_\sigma, \sigma < \rho \), where \( A \) is constant independent of \( z \) and \( \sigma \). These classes remain invariant under conformal transformation if the point at infinity is not involved.

Let \( f(z) \) be given of class \( L(p, \alpha) \) on \( C_\rho \). We shall use the classical map, the inverse of \( z = (w + 1/w)/2 \) which transforms \( E \) into \( E_w : |w| = 1 \), and \( D \) into two regions \( D' \) and \( D'' \) of the \( w \)-plane respectively exterior and interior to \( E_w \). The transform \( F(w) \) of \( f(z) \) is analytic throughout the two regions \( D'_\rho \) and \( D''_\rho \), images of \( D_\rho \), respectively exterior and interior to \( E_w \), and \( F(w) \) is of class \( L(p, \alpha) \) on the images of \( C_\rho \), parts of the boundaries of \( D'_\rho \) and \( D''_\rho \). Thus \( F(w) \) is continuous and hence analytic also on \( E_w \). In the region \( D'_\rho + D''_\rho + E_w \) the function \( F(w) \) is the sum of two components \( F_1(w) \) and \( F_2(w) \) defined by Cauchy's integral extended over sets of Jordan curves in \( D'_\rho \) and \( D''_\rho \) near the boundaries other than \( E_w \) of those respective regions; if \( D'_\rho \) is infinite we add \( F(\infty) \) to the second of these integrals. The components \( F_1(w) \) and \( F_2(w) \) are analytic throughout \( D'_\rho \) plus \( |w| \leq 1 \) and \( D''_\rho \) plus \( |w| \geq 1 \) respectively, of class \( L(p, \alpha) \) on the boundaries.

By the general theory \([1, \text{Theorem 6 and } \S 8]\) there exist families of functions \( F_{\lambda_1}(w) \) and \( F_{\lambda_2}(w) \) analytic throughout \( D' \) plus \( |w| \leq 1 \) and \( D'' \) plus \( |w| \geq 1 \) respectively, hence in particular analytic throughout \( D_w = D' + D'' + E_w \), satisfying

\[
| F_1(w) - F_{\lambda_1}(w) | \leq A_1 e^{-\lambda |w|^{p+\alpha}}, \quad w \text{ on } E_w,
\]
\[
| F_{\lambda_1}(w) | \leq A_2 e^{\lambda |w|^{1-p}/|w|^{p+\alpha}}, \quad w \text{ in } D_w,
\]
\[
| F_2(w) - F_{\lambda_2}(w) | \leq A_1 e^{-\lambda |w|^{p+\alpha}}, \quad w \text{ on } E_w,
\]
\[
| F_{\lambda_2}(w) | \leq A_2 e^{\lambda |w|^{1-p}/|w|^{p+\alpha}}, \quad w \text{ in } D_w.
\]

If we set \( F(w) = F_1(w) + F_2(w), F_\lambda(w) = F_{\lambda_1}(w) + F_{\lambda_2}(w) \), we have consequently

\[
| F(w) - F_\lambda(w) | \leq A_3 e^{-\lambda |w|^{p+\alpha}}, \quad w \text{ on } E_w,
\]
\[
| F_\lambda(w) | \leq A_4 e^{\lambda |w|^{(1-p)/|w|^{p+\alpha}}}, \quad w \text{ in } D_w.
\]

However, \( D_w \) is invariant under the transformation \( w = 1/w \), as also is \( F(w) \), so we have

\[
| F(w) - F_\lambda(1/w) | \leq A_3 e^{-\lambda |w|^{p+\alpha}}, \quad w \text{ on } E_w,
\]
\[
| F_\lambda(1/w) | \leq A_4 e^{\lambda |w|^{(1-p)/|w|^{p+\alpha}}}, \quad w \text{ in } D_w,
\]
\[
|2 F(w) - [F_\lambda(w) + F_\lambda(1/w)]/2 | \leq A_5 e^{-\lambda |w|^{p+\alpha}}, \quad w \text{ on } E_w,
\]
\[
| F_\lambda(w) + F_\lambda(1/w)]/2 | \leq A_4 e^{\lambda |w|^{(1-p)/|w|^{p+\alpha}}}, \quad w \text{ in } D_w.
\]

Under the transformation \( z = (w + 1/w)/2 \), the function

\[
[F_\lambda(w) + F_\lambda(1/w)]/2
\]
is carried into a function \( f_\lambda(z) \) single valued and analytic throughout \( D \); the function \( f_\lambda(z) \) is analytic even on \( E \), since it is analytic in a deleted neighborhood of \( E \) and continuous in the closure of that neighborhood. Thus (1) and (2) follow from (3) and (4).

In the converse direction, inequalities (1) and (2) imply that \( f(z) \) can be extended from \( E \) so as to be analytic throughout \( D_p \), of class \( L(p-1, \alpha) \) on \( C_p \); indeed this fact can be proved by reference to the \( w \)-plane and Theorem 1 of [1], or can be proved by the method of proof of that theorem directly in the given \( z \)-plane, by applying the two-constant theorem to compute degree of convergence on \( C_p \) \((p \geq 0)\) or convergence on \( C_p \) near \( C_p \) \((p < 0)\).

If \( E \) is an arbitrary circular arc (not an entire circumference) and \( D \) is an arbitrary region containing \( E \) and bounded by a finite number of mutually disjoint Jordan curves, the theorem just established applies after a suitable linear transformation of the plane is made, and hence applies in appropriate form before this linear transformation is made. Similarly the conclusion applies to an arbitrary analytic Jordan arc \( E \) and region \( D \) containing \( E \) and bounded by a finite number of mutually disjoint Jordan curves, provided there exists a schlicht one-to-one map of the closure of \( D \) which carries \( E \) into a line segment. But the corresponding theorem where \( E \) is an arbitrary analytic Jordan arc and \( D \) is arbitrary has not yet been established.

In the theorem we have required the boundary of \( D \) to consist of a finite number of mutually disjoint Jordan curves, but this requirement may be weakened. It is sufficient if \( D \) is bounded by a finite number of mutually disjoint continua, none of which is a single point, because that is sufficient for application to \( D_w \) of the previous theory [1]; a succession of conformal maps, say of such a \( D_w \), transforms \( D_w \) into a region bounded by a finite number of mutually disjoint analytic Jordan curves, and transforms \( E_w \) into an analytic Jordan curve.

A previous study exists [2, §8.2] of Problem \( \beta \) for approximation by polynomials, which essentially includes the special case of the present theorem in which \( D \) is bounded by an ellipse with foci \(+1\) and \(-1\). Such approximation by polynomials was first considered by S. Bernstein.

References
