NOTE ON A THEOREM OF FUGLEDE AND PUTNAM

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1. An involution in a ring \( A \) is a mapping \( a \rightarrow a^* (a \in A) \) such that \( a^{**} = a, (a+b)^* = a^* + b^*, (ab)^* = b^*a^* \). An element \( a \in A \) is (1) normal if \( a^*a = aa^* \), (2) self-adjoint if \( a^* = a \), (3) unitary if \( a^*a = aa^* = 1 \) (1 = unity element of \( A \)). We say that “Fuglede’s theorem holds in \( A \)” in case the relations \( a \in A, a \) normal, \( b \in A, ba = ab \), imply \( ba^* = a^*b \); briefly, \( A \) is an FT-ring.

It follows from a theorem of B. Fuglede that the ring \( A \) of all bounded operators in a Hilbert space (hence any adjoint-containing subring thereof) is an FT-ring [3, Theorem I]. For this ring, C. R. Putnam obtained the following generalization [9, Lemma]: if \( a_1, a_2 \) are normal, and \( b a_1 = a_2 b \), then \( b a_1^* = a_2^* b \). A ring with involution, in which the latter theorem holds, will be called a PT-ring.

We denote by \( A_n \) the ring of all \( n \times n \) matrices \( x = (a_{ij}), a_{ij} \in A \), provided with the “conjugate-transpose” involution \( x^* = (a_{ji}) \).

Theorem 1. If \( A_2 \) is an FT-ring, then \( A \) is a PT-ring.

Proof. Suppose \( a_1, a_2 \) are normal elements of \( A \), and \( b a_1 = a_2 b \). Define

\[
x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.
\]

Clearly \( x \) is normal. Moreover,

\[
yx = \begin{pmatrix} 0 & 0 \\ b a_1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 0 \\ a_2 b & 0 \end{pmatrix}
\]

thus \( yx = xy \). Since Fuglede’s theorem holds in \( A_2 \), \( yx^* = x^*y \), in other words \( b a_1^* = a_2^* b \).

Example 1. Let \( A \) be an involutive (i.e. adjoint-containing) ring of bounded operators acting on a Hilbert space \( H \). Then \( A_2 \) is an involutive ring acting on the direct sum of two copies of \( H \). By Fuglede’s theorem, \( A_2 \) is an FT-ring; thus \( A \) is a PT-ring by Theorem 1. This is Putnam’s generalization of the Fuglede theorem [9, Lemma].

In the reverse direction, if \( A \) is a PT-ring, then Fuglede’s theorem

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holds for the diagonal normal elements of \( A_2 \); we omit the obvious proof:

**Theorem 2.** If \( A \) is a PT-ring, and \( a_1, a_2 \in A \) are normal, then the commutant of the normal matrix

\[
x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}
\]

in \( A_2 \) is involutive; that is, the relations \( y \in A_2, yx = xy, \) imply \( yx^* = x^*y \).

A ring \( A \) with involution is said to satisfy the *square root axiom* [6, Chapter VII] in case: given any \( a \in A \), there exists a self-adjoint element \( r \) such that \( r^2 = a^*a \), and such that \( r \) is in the double commutant of \( a^*a \) (that is, the relation \( b(a^*a) - (a^*a)b \) implies \( br = rb \)).

**Examples:** any \( C^* \)-algebra (see [7, Theorem 26A]); the regular ring of a finite AW*-algebra [1, Corollary 6.2]. Suppose \( A \) is a ring satisfying the SR-axiom, and \( a \in A \) is invertible. Write \( u = ar^{-1} \), where \( r \) is the self-adjoint described above; clearly \( u^*u = uu^* = 1 \). The factorization \( a = ur \) is called a “polar decomposition” for \( a \).

**Theorem 3.** Let \( A \) be a PT-ring satisfying the square-root axiom. If \( a_1, a_2 \) are similar normal elements, they are unitarily equivalent.

**Proof.** Suppose \( ba_1b^{-1} = a_2 \). Then \( ba_1 = a_2b \); since \( A \) is a PT-ring, \( ba_1^* = a_2^*b \), thus \( a_1b^* = b^*a_2 \). Let \( b = ur \) be a polar decomposition. Then \( a_1 \) commutes with \( b^*b \); for, \( a_1(b^*b) = (a_1b^*)b = (b^*a_2)b = b^*(a_2b) \)

\( = b^*(ba_1) = (b^*b)a_1 \). Hence \( a_1r = ra_1 \), and \( a_2 = ba_1b^{-1} = (ur)a_1(r^{-1}u^*) \)

\( = ua_br^{-1}u^* = ua_1u^* \).

**Example 2 (Putnam).** If \( A \) is the ring of all bounded operators in a Hilbert space, and \( a_1, a_2 \in A \) are similar normal operators, then \( a_1, a_2 \) are unitarily equivalent by Example 1 and Theorem 3 (see [9, Theorem 1]). The argument works just as well for \( A \) any \( C^* \)-algebra, the point being that the elements implementing the similarity and unitary equivalence are to be drawn from \( A \).

A ring \( A \) with involution is said to possess a *trace* if there exists a mapping \( a \rightarrow \text{tr}(a) \) of \( A \) into some abelian group, such that (1) \( \text{tr}(a+b) = \text{tr}(a) + \text{tr}(b) \), (2) \( \text{tr}(ab) = \text{tr}(ba) \), and (3) \( \sum_1^k \text{tr}(a_i^*a_i) = 0 \) implies \( a_1 = \cdots = a_k = 0 \).

**Theorem 4.** If \( A \) is a ring with involution and trace, then \( A \) is a PT-ring.

**Proof.** Since \( A_n \) also has a trace, defined for a matrix \( x = (a_{ij}) \) by the formula \( \text{tr}(x) = \sum_1^n \text{tr}(a_{ii}) \), it will suffice by Theorem 1 to show that \( A \) is an FT-ring. Suppose \( x \) is normal, and \( yx = xy \). It must be
shown that \( z = yx^* - x^*y \) is 0. We learned the ensuing argument for this from I. Kaplansky. One has

\[
zz^* = yx^*xy^* - yx^*y^*x - x^*yy^*x
\]

\[
= yxx^*y^* - yx^*y^*x - x^*yy^* + x^*yy^*x
\]

\[
= xyy^*y^* - yx^*y^*x - xx^*yy^* + x^*yy^*x.
\]

Since \( \text{tr}(xyx^*y^*) = \text{tr}(yx^*y^*x) \), and \( \text{tr}(xx^*yy^*) = \text{tr}(x^*yy^*x) \), one has \( \text{tr}(zz^*) = 0 \), hence \( z = 0 \).

**Example 3.** Let \( A \) be a commutative ring with involution, such that \( \sum a_i a_i^* = 0 \) implies \( a_1 = \cdots = a_k = 0 \), and set \( \text{tr}(a) = a. \) Then \( A_n \) is a PT-ring by Theorem 4.

**Example 4.** Let \( Q \) be the ring of all real quaternions \( a = \alpha + \beta i + \gamma j + \delta k \), with involution \( a^* = \alpha - \beta i - \gamma j - \delta k. \) One has \( a^*a = a a^* = a^2 + \beta^2 + \gamma^2 + \delta^2 \), so that incidentally every element of \( Q \) is normal. Set \( \text{tr}(a) = \alpha. \) It results from Theorem 4 that \( Q_n \) is a PT-ring. This is Putnam's theorem for finite-dimensional quaternionic Hilbert space, and raises the analogous question for infinite dimension.

**Example 5.** Let \( A \) be a homogeneous AW*-algebra of finite order \( n \), so that \( A = Z_n \), where \( Z \) is the center of \( A \). Let \( C \) be the regular ring of \( A \), \( W \) the regular ring of \( Z \); we may identify \( W \) with the center of \( C \) [1, Theorem 9.2]. Now, \( W \) has the properties in Example 3 [1, Lemma 3.4]; since \( C = W_n \) [2, concluding remark (2)], it follows that \( C \) has a \( W \)-valued trace. Thus \( C \) is a PT-ring. See Theorem 5 for the generalization to \( A \) of finite Type I.

**Lemma.** Suppose \( A \) is the C*-sum of a family \( (A_i) \) of finite AW*-algebras, \( C \) is the regular ring of \( A \), and \( C_i \) is the regular ring of \( A_i \). Then \( C \) is the complete direct sum of the \( C_i \).

**Proof.** According to [5, §2], \( A \) is the set of all families \( a = (a_i) \) with \( a_i \in A \) and \( \|a_i\| \) bounded; the operations in \( A \) are coordinate-wise. One knows from [5] that \( A \) is an AW*-algebra, and is clearly of finite class, so that we may speak of its regular ring \( C \).

Let \( D \) be the complete direct sum of the \( C_i \). That is, \( D \) is the set of all families \( x = (x_i) \) with \( x_i \in C_i \), with the coordinatewise operations. By an easy coordinatewise argument, one sees that \( D \) is regular. It must be shown that \( D = C \).

We may identify \( A \) as an involutive subalgebra of \( D \). We shall prove \( D = C \) by verifying the criterion of [1, §11]. Suppose \( x, y, z \in D \), and \( x^*x + y^*y + z^*z = 1 \). Then \( x_i^*x_i + y_i^*y_i + z_i^*z_i = 1 \) for all \( i \), hence \( x_i, y_i, z_i \in A_i \); since these elements all have norm \( \leq 1 \), one has \( x, y, z \in A \).
Theorem 5. If $A$ is a finite $AW^*$-algebra of Type I, its regular ring $C$ possesses a center-valued trace. In particular, $C$ is a PT-ring.

Proof. Write $A$ as the $C^*$-sum of a family $(A_i)$ of homogeneous algebras, and let $C_i$ be the regular ring of $A_i$. By the Lemma, $C$ is the complete direct sum of the $C_i$. It follows at once that the center $W$ of $C$ is the complete direct sum of the centers $W_i$ of $C_i$. According to Example 5, $C_i$ has a $W_i$-valued trace. Then $(x_i) \to (tr x_i)$ defines a $W$-valued trace on $C$, thus $C$ is a PT-ring by Theorem 4.

It is reasonable to suppose that $C$ is a PT-ring, for any finite $AW^*$-algebra $A$; in any case, since $A_2$ is $AW^*$ with regular ring $C_2$ by [2], it would suffice by Theorem 1 to show that $C$ is an FT-ring.

Corollary. $A$, $C$ as in Theorem 5. If $z_1$, $z_2$ are similar normal elements of $C$, they are unitarily equivalent.

Proof. $C$ is a PT-ring, with square root axiom [1, Corollary 6.2]; quote Theorem 3.

It results from the corollary that if two normal elements are similar via an unbounded element, they are already similar via a bounded (even unitary) element; in particular, a normal bounded element cannot be similar to a normal unbounded element. Normality is essential here, as is shown by the following example due to Jacob Feldman:

Example 6. Let $A$ be the $C^*$-sum of denumerably many copies of the algebra $K_2$ of $2 \times 2$ complex matrices. $A$ may be represented as the algebra of all functions $n \to f(n)$ ($n = 1, 2, 3, \cdots$), with $f(n) \in K_2$, $\|f(n)\|$ bounded, and operations pointwise. Since $K_2$ is its own regular ring, the regular ring $C$ of $A$ is the algebra of all functions $n \to f(n)$ with $f(n) \in K_2$. Consider the functions $f$, $g$, $h \in C$ defined by

$$f(n) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g(n) = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}, \quad h(n) = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$  

Since $h(n)f(n)h(n)^{-1} = g(n)$ for all $n$, one has $hfh^{-1} = g$. Thus $f$ and $g$ are similar in $C$, even though $f$ is bounded (i.e., is an element of $A$) and $g$ is not bounded.

2. More on the regular ring. Throughout, $C$ denotes the regular ring of a finite $AW^*$-algebra $A$ (of unrestricted type).

If $x \in C$, and $RP(x) = 1$, then $x$ is invertible. For, $Cx = C$ [1, Corollary 7.1], so there exists $y \in C$ with $yx = 1$; moreover $LP(x) \sim RP(x) = 1$, hence $LP(x) = 1$ by finiteness, $xC = C$, $xz = 1$ for suitable $z$. Note that an $x \in C$ is invertible if and only if it is left (right) invertible.

If $x \in C$ is invertible, then $x^*$ is invertible, and $(x^*)^{-1} = (x^{-1})^*$; if moreover $x$ is self-adjoint, so is $x^{-1}$. 

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Lemma 1. If $x \in \mathbb{C}$, $x \geq 0$, and $x$ is invertible, then $x^{-1} \geq 0$.

Proof. Say $xy = yx = 1$, and $x = z^*z$ [1, Definition 6.1]. Then $(yz^*)z = 1$ shows that $z$ is invertible (see above remarks), hence $x^{-1} = (z^*z)^{-1} = (z^{-1})(z^{-1})^* \geq 0$.

Lemma 2. Let $a \in A$, $0 \leq a \leq 1$, and suppose $a$ has an inverse in $\mathbb{C}$. Then $a^{-1} \geq 1$.

Proof. Say $ax = xa = 1$; we know $x \geq 0$ from Lemma 1. Write $x = y^2$, $y$ self-adjoint [1, Corollary 6.2]. Since $(ay)y = y(ay) = 1$, $y$ is invertible, and $ay = ya = y^{-1}$. Then $a \leq 1$, $y^*ay \leq y^{-1}$, $yay \leq y^2$, $ay^2 \leq y^2$, $ax \leq x$, $1 \leq x$.

Theorem 6. Suppose $x, y \in \mathbb{C}$, $0 \leq x \leq y$, and $x$ is invertible. Then $y$ is invertible, and $x^{-1} \geq y^{-1} \geq 0$.

Proof. The relation $0 \leq x \leq y$ implies $RP(x) \geq RP(y)$ (see the proof of [1, Corollary 7.6]); by assumption $RP(x) = 1$, hence $RP(y) = 1$, $y$ is invertible. Write $x^{1/2} = wy^{1/2}$ with $w \in A$, $w^*w \leq 1$ [1, Corollary 7.6]. Then $w = (x^{1/2})(y^{1/2})^{-1}$ is invertible in $\mathbb{C}$, hence so is $w^*w$, and $(w^*w)^{-1} = (w^{-1})^*(w^{-1})^{-1} \geq 1$ by Lemma 2. Since $(x^{1/2})^{-1} = (y^{1/2})^{-1}w^{-1}$, one has $x^{-1} = (x^{1/2})^{-1}(x^{1/2})^{-1} = (y^{1/2})^{-1}(w^{-1})(w^{-1})^*(y^{1/2})^{-1} \geq (y^{1/2})^{-1} \cdot (y^{1/2})^{-1} = y^{-1}$. For a similar result of Rellich, see [4, Hilfsatz 4].

Corollary. Suppose $A$ has the property that every increasingly directed family of self-adjoint elements, which is bounded above, has a least upper bound. Then $C$ has the same property.

Proof. For ease of notation, we write the proof for sequences. Suppose $x_i \in \mathbb{C}$ are self-adjoint, $x_1 \leq x_2 \leq x_3 \leq \cdots$, $y \in \mathbb{C}$ is self-adjoint, and $y \leq y$ for all $i$. Adding $-x_i$ throughout, we can assume $0 \leq x_i \leq y$. Then $1 \leq 1 + x_1 \leq 1 + x_2 \leq \cdots \leq 1 + y$, hence by Theorem 6, $1 \geq (1 + x_i)^{-1} \geq (1 + x_2)^{-1} \geq \cdots \geq (1 + y)^{-1} \geq 0$. But $(1 + x_i)^{-1}$ and $(1 + y)^{-1}$ belong to $A$ [1, Lemma 5.1]. Let $a \in A$ be the greatest lower bound of the $(1 + x_i)^{-1}$; one has $0 \leq (1 + y)^{-1} \leq a \leq (1 + x_i)^{-1}$. By Theorem 6, $a$ has an inverse in $C$, and $1 + y \geq a^{-1} \geq 1 + x_i$. Evidently $a^{-1} - 1$ is a least upper bound for the $x_i$. (Example: $A$ any finite $W^*$-algebra; see [8, Theorem 1].)

Lemma. Let $z \in \mathbb{C}$ be normal, and suppose there exists a complex number $\lambda$ such that $z - \lambda$ has an inverse in $A$. Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.

Proof. (We are assuming, so to speak, that the "resolvent set" of $z$ is nonempty.) Suppose $a \in A$, $az = za$. Then $a(z - \lambda) = (z - \lambda)a$, and $z - \lambda$ is normal. Changing notation, assume $z$ invertible, $z^{-1} \in A$,
az = za. Then \( z^{-1}a = az^{-1} \), hence by Fuglede's theorem \( a(z^{-1})^* = (z^{-1})^*a \), \( a(z^*)^{-1} = (z^*)^{-1}a \), \( z^*a = az^* \).

**Theorem 7.** Let \( z \in \mathcal{C} \) be normal, and write \( z = x + iy \) with \( x \) and \( y \) self-adjoint. Suppose there exists a real number \( \alpha \) such that \( x - \alpha \) (or \( y - \alpha \)) has an inverse in \( \mathcal{A} \). Then the relations \( \alpha \in \mathcal{A}, az = za, \) imply \( az^* = z^*a \).

**Proof.** Passing to \( iz \) if necessary, we may suppose that it is \( x - \alpha \) which has a bounded inverse. Then \( (x - \alpha)^{-2} = (x - \alpha)^{-1}(x - \alpha)^{-1} \leq \beta \) for a suitable real number \( \beta > 0 \). By Theorem 6, \( (x - \alpha)^2 \geq 1/\beta > 0 \). Since \( yx = xy \) by normality, and \( z - \alpha = (x - \alpha) + iy \), we have \( (z - \alpha)^* \cdot (z - \alpha) = (x - \alpha)^2 + y^2 \geq (x - \alpha)^2 \geq 1/\beta > 0 \). Hence \( (z - \alpha)^*(z - \alpha) \) is invertible, and \( (z - \alpha)^{-1}(z - \alpha)^{-1} \leq \beta \). Therefore \( (z - \alpha)^{-1} \in \mathcal{A} \) [1, Lemma 5.1]; quote the lemma.

A self-adjoint \( x \in \mathcal{C} \) is **semi-bounded** in case there exists a real number \( \beta \) such that either \( x \leq \beta \) or \( x \geq \beta \). For instance if \( x \in \mathcal{A} \) is self-adjoint, then \( x \leq \| x \| \). If \( x \) is semi-bounded, say \( x \geq \beta \), then setting \( \alpha = \beta - 1 \), one has \( x - \alpha \geq 1 \), hence \( x - \alpha \) has a bounded inverse (Theorem 6, and Lemma 5.1 of [1]). Thus:

**Corollary.** Let \( z \in \mathcal{C} \) be normal, and write \( z = x + iy \), with \( x \) and \( y \) self-adjoint. Suppose either \( x \) or \( y \) is semi-bounded. Then the relations \( x \in \mathcal{A}, az = za, \) imply \( az^* = z^*a \).

If \( \mathcal{A} \) has a trace (e.g. if \( \mathcal{A} \) is Type I, or is a finite \( \mathcal{W}^*_\ast \)-algebra), it is clear that the relations \( x \in \mathcal{A}, a^*a \leq aa^*, \) imply \( a^*a = aa^* \). We do not know if every finite \( \mathcal{A} \)-algebra \( \mathcal{A} \) has this property, but whenever \( \mathcal{A} \) does, so does \( \mathcal{C} \):

**Theorem 8.** Suppose the relations \( x \in \mathcal{C}, x^*x \leq xx^* \), imply \( a^*a = aa^* \). Then the relations \( x \in \mathcal{C}, x^*x \leq xx^* \), imply \( x^*x = xx^* \).

**Proof.** Suppose \( x^*x \leq xx^* \). Write \( x = ur, r \geq 0, u \) unitary [1, Corollary 7.4]. Then \( x^*x = r^2 \), and \( xx^* = ur^2u^* = u(x^*x)u^* \). Setting \( s = x^*x, t = xx^* \), we have \( 0 \leq s \leq t \), and \( s, t \) are unitarily equivalent. Set \( b = (1 + s)^{-1}, c = (1 + t)^{-1} \); clearly \( b, c \) are unitarily equivalent, in fact \( usu^* = t \) yields \( ubu^* = c \). Moreover \( b \geq c \) by Theorem 6, and \( b, c \in \mathcal{A} \) [1, Lemma 5.1]. Set \( a = b^{1/2}u^* \). Then \( aa^* = b^{1/2}u^*ub^{1/2} = b \geq c \geq ubu^* = a^*a \). By the hypothesis on \( \mathcal{A} \), \( aa^* = a^*a \), hence \( b = c \), and this leads to \( s = t \).

**Corollary 1.** Suppose the relations \( a \in \mathcal{A}, a^*a \leq aa^* \), imply \( a^*a = aa^* \). If \( x \in \mathcal{C} \), and \( x \) commutes with \( x^*x \), then \( x \) is normal.

**Proof.** By assumption \( xx^*x = x^*xx \). Right-multiplying by \( x^* \),
$xx*xx*=x*x_{xx*}$. Setting $r=x*x$, $s=xx*$, we have $r \geq 0$, $s \geq 0$, and $s^2=rs$. In particular $rs$ is self-adjoint, so that $rs=sr$. Hence by uniqueness of positive square roots, $s=(s^2)^{1/2}=(rs)^{1/2}=r^{1/2}s^{1/2}$ (see [1, remarks following Definition 6.3]). Then $0 \leq (r^{1/2} - s^{1/2})^2 = r - 2r^{1/2}s^{1/2} + s = r - 2s + s = r - s$, thus $0 \leq s \leq r$. That is, $xx* \leq x*x$, hence $xx* = x*x$ by Theorem 8.

**Remark.** In an infinite algebra $B$, choose $x \in B$ with $x*x=1$ but $xx* \neq 1$. Then $x$ commutes with $x*x$, but is not normal.

**Corollary 2.** Suppose the relations $a \in A$, $a*a \leq aa*$, imply $a*a = aa*$. Then every triangular normal matrix in $C_n$ is diagonal.

**Proof.** Suppose e.g. $n = 3$, and

$$x = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

is a normal element of $C_3$. From the 1-1 position in the relation $x*xz=zx*$, one has $a*a = aa* + bb* + cc*$, thus $a*a - aa* = bb* + cc* \geq 0$ [1, Theorem 6.1], $aa* \leq a*a$. By Theorem 8, $aa* = a*a$, hence $b = c = 0$ [1, Lemma 3.4]. Inspection now of the 2-2 position similarly yields $e = 0$. The case for general $n$ is an obvious induction.

**Remark.** If $B$ is an infinite algebra, there exists a normal (even unitary) matrix

$$x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

in $B_2$ with $b \neq 0$. For, choose a partial isometry $a \in A$ with $a*a=1$, $aa*=e \neq 1$, and set $b = 1 - e$, $c = a*$.

**Addenda.** (1) I am indebted to J. Dixmier for calling my attention to the references [4] and [8].

(2) Recently M. Rosenblum has given a beautiful proof of the Fuglede-Putnam theorem; for bounded operators, the proof is non-spatial (see [10]).

**References**


A CORRECTION AND IMPROVEMENT OF A THEOREM ON ORDERED GROUPS

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In this note the notation and terminology of [1] will be used throughout. In particular, G will always denote an o-group with well ordered rank. Let P be the multiplicative group of positive rational numbers, and let R be the additive group of real numbers. In [1] the proofs of Theorems 2 and 3 are incorrect. This is a result of the careless formulation of the theorems by the author. Consider the following properties of G.

(1) Each component Gγ/Gγ of G has its group of o-automorphisms isomorphic to a subgroup Pγ of P.

(2) Each component Gγ/Gγ of G is o-isomorphic to a subgroup Dγ of R, and the only o-automorphisms of Dγ are multiplications by some elements of P.

(3) For each pair α ∈ A and γ ∈ Γ, there exists a pair m, n of positive integers such that ngα ≡ mg mod Gγ for all g in Gγ.

The statements of Theorems 2 and 3 include the hypothesis (1), but (2) and (3) are actually used in the proofs. Clearly (2) implies (1).

Lemma. (a) (2) is independent of the particular choice of Dγ. (b) (2) implies (3). (c) (1) does not imply (2) or (3).

Proof. (a) Let σ be an o-isomorphism of the subgroup A of R onto the subgroup B of R, and suppose that the only o-automorphisms of A are multiplications by some elements of P. If β is an o-automor-