NOTE ON A THEOREM OF FUGLEDE AND PUTNAM

S. K. BERBERIAN

1. An involution in a ring $A$ is a mapping $a \rightarrow a^*$ ($a \in A$) such that $a^{**} = a$, $(a + b)^* = a^* + b^*$, $(ab)^* = b^*a^*$. An element $a \in A$ is (1) normal if $a^*a = aa^*$, (2) self-adjoint if $a^* = a$, (3) unitary if $a^*a = aa^* = 1$ ($1 = $ unity element of $A$). We say that “Fuglede's theorem holds in $A$” in case the relations $a \in A$, $a$ normal, $b \in A$, $ba = ab$, imply $ba^* = a^*b$; briefly, $A$ is an FT-ring.

It follows from a theorem of B. Fuglede that the ring $A$ of all bounded operators in a Hilbert space (hence any adjoint-containing subring thereof) is an FT-ring [3, Theorem 1]. For this ring, C. R. Putnam obtained the following generalization [9, Lemma]: if $a_1$, $a_2$ are normal, and $ba_1 = a_2b$, then $ba_1^* = a_2^*b$. A ring with involution, in which the latter theorem holds, will be called a PT-ring.

We denote by $A_n$ the ring of all $n \times n$ matrices $x = (a_{ij})$, $a_{ij} \in A$, provided with the “conjugate-transpose” involution $x^* = (a_{ji}^*)$.

**Theorem 1.** If $A_2$ is an FT-ring, then $A$ is a PT-ring.

**Proof.** Suppose $a_1$, $a_2$ are normal elements of $A$, and $ba_1 = a_2b$. Define

$$x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$ 

Clearly $x$ is normal. Moreover,

$$yx = \begin{pmatrix} 0 & 0 \\ ba_1 & 0 \end{pmatrix}, \quad xy = \begin{pmatrix} 0 & 0 \\ a_2b & 0 \end{pmatrix}$$

thus $yx = xy$. Since Fuglede's theorem holds in $A_2$, $yx^* = x^*y$, in other words $ba_1^* = a_2^*b$.

**Example 1.** Let $A$ be an involutive (i.e. adjoint-containing) ring of bounded operators acting on a Hilbert space $H$. Then $A_2$ is an involutive ring acting on the direct sum of two copies of $H$. By Fuglede's theorem, $A_2$ is an FT-ring; thus $A$ is a PT-ring by Theorem 1. This is Putnam's generalization of the Fuglede theorem [9, Lemma]. The argument extends easily to cover the case that $a_1$, $a_2$ are possibly unbounded. The result then reads: if $ba_1 \subset a_2b$ then $ba_1^* \subset a_2^*b$.

In the reverse direction, if $A$ is a PT-ring, then Fuglede's theorem

Presented to the Society, August 27, 1958; received by the editors July 28, 1958.
holds for the diagonal normal elements of $A_2$; we omit the obvious proof:

**Theorem 2.** If $A$ is a $PT$-ring, and $a_1, a_2 \in A$ are normal, then the commutant of the normal matrix

$$x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

in $A_2$ is involutive; that is, the relations $y \in A_2, yx = xy$, imply $yx^* = x^*y$.

A ring $A$ with involution is said to satisfy the *square root axiom* ([6, Chapter VII] in case: given any $a \in A$, there exists a self-adjoint element $r$ such that $r^2 = a^*a$, and such that $r$ is in the double commutant of $a^*a$ (that is, the relation $b(a^*a) = (a^*a)b$ implies $br = rb$).

**Examples:** any $C^*$-algebra (see [7, Theorem 26A]); the regular ring of a finite AW*-algebra [1, Corollary 6.2]. Suppose $A$ is a ring satisfying the SR-axiom, and $a \in A$ is invertible. Write $u = ar^{-1}$, where $r$ is the self-adjoint described above; clearly $u^*u = uu^* = 1$. The factorization $a = ur$ is called a “polar decomposition” for $a$.

**Theorem 3.** Let $A$ be a $PT$-ring satisfying the square-root axiom. If $a_1, a_2$ are similar normal elements, they are unitarily equivalent.

**Proof.** Suppose $ba_1b^{-1} = a_2$. Then $ba_1 = a_2b$; since $A$ is a $PT$-ring, $ba_1^* = a_2^*b$, thus $a_1b^* = b^*a_2$. Let $b = ur$ be a polar decomposition. Then $a_1$ commutes with $b^*b$; for, $a_1(b^*b) = (a_1b^*)b = (b^*a_2)b = b^*(a_2b) = b^*(ba_1) = (b^*b)a_1$. Hence $a_1r = ra_1$, and $a_2 = ba_1b^{-1} = (ur)a_1(r^{-1}u^*) = ua_1ru^{-1}u^* = ua_1u^*$.

**Example 2 (Putnam).** If $A$ is the ring of all bounded operators in a Hilbert space, and $a_1, a_2 \in A$ are similar normal operators, then $a_1, a_2$ are unitarily equivalent by Example 1 and Theorem 3 (see [9, Theorem 1]). The argument works just as well for $A$ any $C^*$-algebra, the point being that the elements implementing the similarity and unitary equivalence are to be drawn from $A$.

A ring $A$ with involution is said to possess a *trace* if there exists a mapping $a \rightarrow \text{tr}(a)$ of $A$ into some abelian group, such that (1) $\text{tr}(a + b) = \text{tr}(a) + \text{tr}(b)$, (2) $\text{tr}(ab) = \text{tr}(ba)$, and (3) $\sum a_i a_i = 0$ implies $a_1 = \cdots = a_k = 0$.

**Theorem 4.** If $A$ is a ring with involution and trace, then $A$ is a $PT$-ring.

**Proof.** Since $A_n$ also has a trace, defined for a matrix $x = (a_{ij})$ by the formula $\text{tr}(x) = \sum a_{ii}$, it will suffice by Theorem 1 to show that $A$ is an FT-ring. Suppose $x$ is normal, and $yx = xy$. It must be
shown that $z = yx^* - x^*y$ is 0. We learned the ensuing argument for this from I. Kaplansky. One has

$$zz^* = yx^*xy^* - yx^*y^*x - xy^*xy^* + x^*yy^*x$$

$$= yxx^*y^* - xx^*yy^* + x^*yy^*x$$

Since $\text{tr}(yx^*y^*) = \text{tr}(yx^*y^*x)$, and $\text{tr}(xx^*yy^*) = \text{tr}(x^*yy^*x)$, one has $\text{tr}(zz^*) = 0$, hence $z = 0$.

Example 3. Let $A$ be a commutative ring with involution, such that $\sum a_i^* a_i = 0$ implies $a_1 = \cdots = a_k = 0$, and set $\text{tr}(a) = a$. Then $A_n$ is a PT-ring by Theorem 4.

Example 4. Let $Q$ be the ring of all real quaternions $a = \alpha + \beta i + \gamma j + \delta k$, with involution $a^* = \alpha - \beta i - \gamma j - \delta k$. One has $a^*a = aa^* = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$, so that incidentally every element of $Q$ is normal. Set $\text{tr}(a) = \alpha$. It results from Theorem 4 that $Q_n$ is a PT-ring. This is Putnam’s theorem for finite-dimensional quaternionic Hilbert space, and raises the analogous question for infinite dimension.

Example 5. Let $A$ be a homogeneous AW*-algebra of finite order $n$, so that $A = Z_n$, where $Z$ is the center of $A$. Let $C$ be the regular ring of $A$, $W$ the regular ring of $Z$; we may identify $W$ with the center of $C$ [1, Theorem 9.2]. Now, $W$ has the properties in Example 3 [1, Lemma 3.4]; since $C = W_n$ [2, concluding remark (2)], it follows that $C$ has a $W$-valued trace. Thus $C$ is a PT-ring. See Theorem 5 for the generalization to $A$ of finite Type I.

Lemma. Suppose $A$ is the $C^*$-sum of a family $(A_i)$ of finite AW*-algebras, $C$ is the regular ring of $A$, and $C_i$ is the regular ring of $A_i$. Then $C$ is the complete direct sum of the $C_i$.

Proof. According to [5, §2], $A$ is the set of all families $a = (a_i)$ with $a_i \in A_i$ and $\|a_i\|$ bounded; the operations in $A$ are coordinate-wise. One knows from [5] that $A$ is an AW*-algebra, and is clearly of finite class, so that we may speak of its regular ring $C$.

Let $D$ be the complete direct sum of the $C_i$. That is, $D$ is the set of all families $x = (x_i)$ with $x_i \in C_i$, with the coordinatewise operations. By an easy coordinatewise argument, one sees that $D$ is regular. It must be shown that $D = C$.

We may identify $A$ as an involutive subalgebra of $D$. We shall prove $D = C$ by verifying the criterion of [1, §11]. Suppose $x, y, z \in D$, and $x^*x + y^*y + z^*z = 1$. Then $x_i^*x_i + y_i^*y_i + z_i^*z_i = 1$ for all $i$, hence $x_i, y_i, z_i \in A_i$; since these elements all have norm $\leq 1$, one has $x, y, z \in A$. 

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Theorem 5. If $A$ is a finite AW*-algebra of Type I, its regular ring $C$ possesses a center-valued trace. In particular, $C$ is a PT-ring.

Proof. Write $A$ as the $C^*$-sum of a family $(A_i)$ of homogeneous algebras, and let $C_i$ be the regular ring of $A_i$. By the Lemma, $C$ is the complete direct sum of the $C_i$. It follows at once that the center $W$ of $C$ is the complete direct sum of the centers $W_i$ of $C_i$. According to Example 5, $C_i$ has a $W_i$-valued trace. Then $(x_i) \rightarrow (\text{tr} x_i)$ defines a $W$-valued trace on $C$, thus $C$ is a PT-ring by Theorem 4.

It is reasonable to suppose that $C$ is a PT-ring, for any finite AW*-algebra $A$; in any case, since $A_2$ is AW* with regular ring $C_2$ by [2], it would suffice by Theorem 1 to show that $C$ is an FT-ring.

Corollary. $A, C$ as in Theorem 5. If $z_1, z_2$ are similar normal elements of $C$, they are unitarily equivalent.

Proof. $C$ is a PT-ring, with square root axiom [1, Corollary 6.2]; quote Theorem 3.

It results from the corollary that if two normal elements are similar via an unbounded element, they are already similar via a bounded (even unitary) element; in particular, a normal bounded element cannot be similar to a normal unbounded element. Normality is essential here, as is shown by the following example due to Jacob Feldman:

Example 6. Let $A$ be the $C^*$-sum of denumerably many copies of the algebra $K_2$ of $2 \times 2$ complex matrices. $A$ may be represented as the algebra of all functions $n \rightarrow f(n) \ (n = 1, 2, 3, \ldots)$, with $f(n) \in K_2$, $\|f(n)\|$ bounded, and operations pointwise. Since $K_2$ is its own regular ring, the regular ring $C$ of $A$ is the algebra of all functions $n \rightarrow f(n)$ with $f(n) \in K_2$. Consider the functions $f, g, h \in C$ defined by

$$f(n) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g(n) = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}, \quad h(n) = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$ 

Since $h(n)f(n)h(n)^{-1} = g(n)$ for all $n$, one has $hf^{-1} = g$. Thus $f$ and $g$ are similar in $C$, even though $f$ is bounded (i.e., is an element of $A$) and $g$ is not bounded.

2. More on the regular ring. Throughout, $C$ denotes the regular ring of a finite AW*-algebra $A$ (of unrestricted type).

If $x \in C$, and $RP(x) = 1$, then $x$ is invertible. For, $Cx = C$ [1, Corollary 7.1], so there exists $y \in C$ with $yx = 1$; moreover $LP(x) \sim RP(x) = 1$, hence $LP(x) = 1$ by finiteness, $xC = C, xz = 1$ for suitable $z$. Note that an $x \in C$ is invertible if and only if it is left (right) invertible.

If $x \in C$ is invertible, then $x^*$ is invertible, and $(x^*)^{-1} = (x^{-1})^*$; if moreover $x$ is self-adjoint, so is $x^{-1}$.
Lemma 1. If \( x \in C, \ x \geq 0, \) and \( x \) is invertible, then \( x^{-1} \geq 0. \)

Proof. Say \( xy = yx = 1, \) and \( x = z^*z \) [1, Definition 6.1]. Then \((yz^*)z = 1\) shows that \( z \) is invertible (see above remarks), hence \( x^{-1} = (z^*z)^{-1} = (z^{-1})(z^{-1})^* \geq 0.\)

Lemma 2. Let \( a \in A, \ 0 \leq a \leq 1, \) and suppose \( a \) has an inverse in \( C. \) Then \( a^{-1} \geq 1. \)

Proof. Say \( ax = xa = 1; \) we know \( x \geq 0 \) from Lemma 1. Write \( x = y^2, \) \( y \) self-adjoint [1, Corollary 6.2]. Since \((ay)y = y(ya) = 1, \) \( y \) is invertible, and \( ay = ya = y^{-1}. \) Then \( a \leq 1, \) \( y^*ay \leq y^*y, \) \( yay \leq y^2, \) \( ay^2 \leq y^2, \) \( ax \leq x, \ 1 \leq x. \)

Theorem 6. Suppose \( x, y \in C, \ 0 \leq x \leq y, \) and \( x \) is invertible. Then \( y \) is invertible, and \( x^{-1} \leq y^{-1} \geq 0. \)

Proof. The relation \( 0 \leq x \leq y \) implies \( RP(x) \leq RP(y) \) (see the proof of [1, Corollary 7.6]); by assumption \( RP(x) = 1, \) hence \( RP(y) = 1, \) \( y \) is invertible. Write \( x^{1/2} = wy^{1/2} \) with \( w \in A, \) \( w^*w \leq 1 \) [1, Corollary 7.6]. Then \( w = (x^{1/2})(y^{1/2})^{-1} \) is invertible in \( C, \) hence so is \( w^*w, \) and \((w^*w)^{-1} = (w^{-1})(w^{-1})^* \geq 1 \) by Lemma 2. Since \((x^{1/2})^{-1} = (y^{1/2})^{-1}w^{-1}, \) one has \( x^{-1} = (x^{1/2})^{-1}(x^{1/2})^{-1} = (y^{1/2})^{-1}(w^{-1})(w^{-1})^* \) \((y^{1/2})^{-1} \cdot (y^{1/2})^{-1} = y^{-1}. \) For a similar result of Rellich, see [4, Hilfsatz 4].

Corollary. Suppose \( A \) has the property that every increasingly directed family of self-adjoint elements, which is bounded above, has a least upper bound. Then \( C \) has the same property.

Proof. For ease of notation, we write the proof for sequences. Suppose \( x_i \in C \) are self-adjoint, \( x_1 \leq x_2 \leq x_3 \leq \cdots, \) \( y \in C \) is self-adjoint, and \( x_i \leq y \) for all \( i. \) Adding \(-x_1 \) throughout, we can assume \( 0 \leq x_i \leq y. \) Then \( 1 \leq 1 + x_1 \leq 1 + x_2 \leq \cdots \leq 1 + y, \) hence by Theorem 6, \( 1 \geq (1 + x_1)^{-1} \geq (1 + x_2)^{-1} \geq \cdots \geq (1 + y)^{-1} \geq 0. \) But \((1 + x_i)^{-1} \) and \((1 + y)^{-1} \) belong to \( A \) [1, Lemma 5.1]. Let \( a \in A \) be the greatest lower bound of the \((1 + x_i)^{-1}; \) one has \( 0 \leq (1 + y)^{-1} \leq a \leq (1 + x_i)^{-1}. \) By Theorem 6, \( a \) has an inverse in \( C, \) and \( 1 + y \geq a^{-1} \geq 1 + x_i. \) Evidently \( a^{-1} - 1 \) is a least upper bound for the \( x_i. \) (Example: \( A \) any finite \( W^* \)-algebra; see [8, Theorem 1].)

Lemma. Let \( z \in C \) be normal, and suppose there exists a complex number \( \lambda \) such that \( z - \lambda \) has an inverse in \( A. \) Then the relations \( a \in A, \) \( az = za, \) imply \( az^* = z^*a. \)

Proof. (We are assuming, so to speak, that the "resolvent set" of \( z \) is nonempty.) Suppose \( a \in A, \) \( az = za. \) Then \( a(z - \lambda) = (z - \lambda)a, \) and \( z - \lambda \) is normal. Changing notation, assume \( z \) invertible, \( z^{-1} \in A, \)
az = za. Then $z^{-1}a = az^{-1}$, hence by Fuglede's theorem $a(z^{-1})^* = (z^{-1})^*a$, $a(z^*)^{-1} = (z^*)^{-1}a$, $z^*a = az^*$.

**Theorem 7.** Let $z \in C$ be normal, and write $z = x + iy$ with $x$ and $y$ self-adjoint. Suppose there exists a real number $\alpha$ such that $x - \alpha$ (or $y - \alpha$) has an inverse in $A$. Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.

**Proof.** Passing to $iz$ if necessary, we may suppose that it is $x - \alpha$ which has a bounded inverse. Then $(x - \alpha)^{-2} = (x - \alpha)^{-1}(x - \alpha)^{-1} \leq \beta$ for a suitable real number $\beta > 0$. By Theorem 6, $(x - \alpha)^2 \geq 1/\beta > 0$. Since $yx = xy$ by normality, and $z = x - \alpha + iy$, we have $(z - \alpha)^* \cdot (z - \alpha) = (x - \alpha)^2 + y^2 \geq (x - \alpha)^2 \geq 1/\beta > 0$. Hence $(z - \alpha)^*(z - \alpha)$ is invertible, and $(z - \alpha)^{-1}(z - \alpha)^{-1} \leq \beta$. Therefore $(z - \alpha)^{-1} \in A$ [1, Lemma 5.1]; quote the lemma.

A self-adjoint $x \in C$ is **semi-bounded** in case there exists a real number $\beta$ such that either $x \leq \beta$ or $x \geq \beta$. For instance if $x \in A$ is self-adjoint, then $x \leq \|x\|$. If $x$ is semi-bounded, say $x \geq \beta$, then setting $\alpha = \beta - 1$, one has $x - \alpha \geq 1$, hence $x - \alpha$ has a bounded inverse (Theorem 6, and Lemma 5.1 of [1]). Thus:

**Corollary.** Let $z \in C$ be normal, and write $z = x + iy$, with $x$ and $y$ self-adjoint. Suppose either $x$ or $y$ is semi-bounded. Then the relations $a \in A$, $az = za$, imply $az^* = z^*a$.

If $A$ has a trace (e.g. if $A$ is Type I, or is a finite $W^*$-algebra), it is clear that the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. We do not know if every finite AW*-algebra $A$ has this property, but whenever $A$ does, so does $C$.

**Theorem 8.** Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. Then the relations $x \in C$, $x^*x \leq xx^*$, imply $x^*x = xx^*$.

**Proof.** Suppose $x^*x \leq xx^*$. Write $x = ur$, $r \geq 0$, $u$ unitary [1, Corollary 7.4]. Then $x^*x = r^2$, and $xx^* = ur^2u^* = u(x^*x)u^*$. Setting $s = x^*x$, $t = xx^*$, we have $0 \leq s \leq t$, and $s$, $t$ are unitarily equivalent. Set $b = (1 + s)^{-1}$, $c = (1 + t)^{-1}$; clearly $b$, $c$ are unitarily equivalent, in fact $usu^* = t$ yields $ubu^* = c$. Moreover $b \geq c$ by Theorem 6, and $b$, $c \in A$ [1, Lemma 5.1]. Set $a = b^{1/2}u^*$. Then $aa^* = b^{1/2}u^*ub^{1/2} = b \geq c = ubu^* = a^*a$. By the hypothesis on $A$, $aa^* = a^*a$, hence $b = c$, and this leads to $s = t$.

**Corollary 1.** Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. If $x \in C$, and $x$ commutes with $x^*x$, then $x$ is normal.

**Proof.** By assumption $xx^*x = x^*xx$. Right-multiplying by $x^*$,
$xx*xx*=x*xxx*$. Setting $r=x*x$, $s=xx*$, we have $r \geq 0$, $s \geq 0$, and $s^2=rs$. In particular $rs$ is self-adjoint, so that $rs=rs$. Hence by uniqueness of positive square roots, $s=(s^2)^{1/2}=(rs)^{1/2}=r^{1/2}s^{1/2}$ (see [1, remarks following Definition 6.3]). Then $0 \leq (r^{1/2} - s^{1/2})^2 = r - 2r^{1/2}s^{1/2} + s = r - 2s + s = r - s$, thus $0 \leq s \leq r$. That is, $xx* \leq x*x$, hence $xx*=x*x$ by Theorem 8.

Remark. In an infinite algebra $B$, choose $x \in B$ with $x*x=1$ but $xx* \neq 1$. Then $x$ commutes with $x*x$, but is not normal.

Corollary 2. Suppose the relations $a \in A$, $a^*a \leq aa^*$, imply $a^*a = aa^*$. Then every triangular normal matrix in $C_n$ is diagonal.

Proof. Suppose e.g. $n = 3$, and

$$z = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

is a normal element of $C_3$. From the 1-1 position in the relation $z^*z=zz^*$, one has $a^*a = aa^* + bb^* + cc^*$, thus $a^*a - aa^* = bb^* + cc^* \geq 0$ [1, Theorem 6.1], $aa^* \leq a^*a$. By Theorem 8, $aa^* = a^*a$, hence $b = c = 0$ [1, Lemma 3.4]. Inspection now of the 2-2 position similarly yields $e = 0$. The case for general $n$ is an obvious induction.

Remark. If $B$ is an infinite algebra, there exists a normal (even unitary) matrix

$$x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

in $B_2$ with $b \neq 0$. For, choose a partial isometry $a \in A$ with $a^*a = 1$, $aa^* = e \neq 1$, and set $b = 1 - e$, $c = a^*$.

Addenda. (1) I am indebted to J. Dixmier for calling my attention to the references [4] and [8].

(2) Recently M. Rosenblum has given a beautiful proof of the Fuglede-Putnam theorem; for bounded operators, the proof is non-spatial (see [10]).

References


State University of Iowa

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A CORRECTION AND IMPROVEMENT OF A THEOREM ON ORDERED GROUPS

PAUL CONRAD

In this note the notation and terminology of [1] will be used throughout. In particular, $G$ will always denote an $o$-group with well ordered rank. Let $P$ be the multiplicative group of positive rational numbers, and let $R$ be the additive group of real numbers. In [1] the proofs of Theorems 2 and 3 are incorrect. This is a result of the careless formulation of the theorems by the author. Consider the following properties of $G$.

1. Each component $G^\gamma/G_\gamma$ of $G$ has its group of $o$-automorphisms isomorphic to a subgroup $P_\gamma$ of $P$.
2. Each component $G^\gamma/G_\gamma$ of $G$ is $o$-isomorphic to a subgroup $D_\gamma$ of $R$, and the only $o$-automorphisms of $D_\gamma$ are multiplications by some elements of $P$.
3. For each pair $\alpha \in \mathfrak{A}$ and $\gamma \in \Gamma$, there exists a pair $m, n$ of positive integers such that $ng\alpha \equiv mg \mod G_\gamma$ for all $g$ in $G^\gamma$.

The statements of Theorems 2 and 3 include the hypothesis (1), but (2) and (3) are actually used in the proofs. Clearly (2) implies (1).

**Lemma.** (a) (2) is independent of the particular choice of $D_\gamma$. (b) (2) implies (3). (c) (1) does not imply (2) or (3).

**Proof.** (a) Let $\sigma$ be an $o$-isomorphism of the subgroup $A$ of $R$ onto the subgroup $B$ of $R$, and suppose that the only $o$-automorphisms of $A$ are multiplications by some elements of $P$. If $\beta$ is an $o$-automor---

Received by the editors August 1, 1958.