CHARACTERIZATION OF SOME ELEMENTARY
TRANSFORMATIONS

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1. Introduction. The purpose of this paper is to study some elementary transformations of surfaces embedded in a 3-dimensional Euclidean space $E^3$. This will be developed analogously to the following theorem [1]:

**Translation Theorem.** Given two closed orientable surfaces $S, \bar{S}$ and a homeomorphism $h: S \rightarrow \bar{S}$ such that: (1) each line joining corresponding points is parallel to a fixed direction $E$, (2) the mean curvatures at corresponding points are equal; moreover $S, \bar{S}$ are assumed not to contain pieces of cylinders in $E$-direction. Then $h$ is a translation.

All surfaces mentioned will be of class $C^2$. The notations in [1] will be adopted except that German letters will be replaced by corresponding capital English ones. For example: $X, N, H$ and $dA$ are respectively used to denote the position vector, the unit vector along inward normal direction, the mean curvature and the surface element of a surface $S$. When a second surface $\bar{S}$ is mentioned, the corresponding quantities are represented by the same letters with bars above them. As in [1], the following formulas will be used:

\begin{align}
    (1.1) & \quad dX \times dX = 2NdA, \\
    (1.2) & \quad dX \times dN = -2HNdA.
\end{align}

A closed nonself-intersecting surface $S$ is said to be convex with respect to a given point $0$, (1) if every straight line through $0$ meets $S$ at no point, at one point of contact or at two distinct points, (2) if there is a differentiable homeomorphism $f: S \rightarrow S$ such that each straight line joining corresponding points passes through $0$.

We intend to prove the following theorems:

**Theorem 1.** Given two closed orientable surfaces $S, \bar{S}$ and a differentiable homeomorphism $h: S \rightarrow \bar{S}$ such that: (1) each straight line $P\bar{P}$ joining the corresponding points $P$ and $\bar{P}$ passes through a fixed point $0$; (2) with $0$ as origin, the quantities $X, \bar{X}, H, \bar{H}$ are related to each other either by (i) $\bar{H}X = HX$ throughout $S$ and $\bar{S}$ or by (ii) $\bar{H}X = -HX$ throughout $S$ and $\bar{S}$. Moreover $S, \bar{S}$ are assumed not to contain pieces of cones with vertex $0$. Then $h$ is a homothetic transformation with center $0$ and with a
positive or negative constant of proportionality according as (i) or (ii) holds.

Theorem 2. Given two closed orientable surfaces $S, \overline{S}$ and a differentiable homeomorphism $h: S \to \overline{S}$, such that: (1) each segment $ PP $ joining the corresponding points $P$ and $\overline{P}$ subtends a constant angle $POP$ about a fixed point $0$, (2) with $0$ as origin, $HX$ and $\overline{HX}$ are equal in magnitude. Moreover, $S, \overline{S}$ are assumed not to contain pieces of cones with vertex $0$. Then $h$ is a similarity with $0$ as center of similitude.

Theorem 3. Given two closed orientable surfaces $S, \overline{S}$ and a differentiable homeomorphism $h: S \to \overline{S}$ such that: (1) each straight line $ PP $ joining corresponding points $P$ and $\overline{P}$ passes through a fixed point $0$; (2) with $0$ as origin, the quantities $X, \overline{X}, H, \overline{H}$ are related to each other either by (i) $\overline{HHX} = -(H+2X \cdot N/X \cdot X)X$ throughout $S$ and $\overline{S}$; or by (ii) $\overline{HHX} = (H+2X \cdot N/X \cdot X)X$ throughout $S$ and $\overline{S}$. Moreover $S, \overline{S}$ are assumed not to contain either pieces of cones with vertex $0$ or the point $0$ itself. Then $h$ is an inversion with center $0$ and with real or pure imaginary radius of inversion according as (i) or (ii) holds.

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2. Proofs of theorems.

Proof of Theorem 1. Assume $\overline{HX} = HX$. Write $X = kX$, where $k = H/\overline{H}$.

Case 1. $0 \in S$ and $0 \in \overline{S}$. Then $k \neq 0, \infty$.

\[ dX \times d\overline{X} = (kdX + Xdk) \times (kdX + Xdk) = k^2(dX \times dX) + 2k(Xdk \times dX). \]

By (1.1), we have

\[ \overline{NdA} = k^2NdA + k(Xdk \times dX), \]

whose scalar product with $\overline{HX} (=HX)$ gives

(2.1) \[ (X \cdot \overline{N})\overline{HdA} = k^2(X \cdot N)HdA. \]

Let $a = (N \times X) \cdot dX, b = (\overline{N} \times X) \cdot dX$ and note that $ddX = 0$, then by use of (1.1) and (1.2), we obtain

\[ da = 2(X \cdot N)HdA + 2dA, \]
\[ db = - X \cdot (d\overline{N} \times dX) + 2(\overline{N} \cdot N)dA \]
\[ = (2/k^2)(X \cdot \overline{N})\overline{HdA} + 2(\overline{N} \cdot N)dA \]

since
\[ \vec{X} \cdot (d\vec{N} \times d\vec{X}) = k^2 \vec{X} \cdot (d\vec{N} \times dX). \]

Hence, by (2.1) we have

(2.2) \[ d(a - b)/2 = (1 - \vec{N} \cdot \vec{N})dA. \]

From Stokes' Theorem, it is evident that

\[ \int \int_S (1 - \vec{N} \cdot \vec{N})dA = 0. \]

Since \( 1 - \vec{N} \cdot \vec{N} \geq 0 \), and \( dA \) always keeps the same sign, we have

\[ 1 - \vec{N} \cdot \vec{N} = 0 \]

and therefore

\[ \vec{N} = \vec{N}. \]

Moreover, since \( \vec{N} \cdot d\vec{X} = 0, \vec{N} \cdot dX = 0 \), we have

\[ (\vec{N} \cdot X)dk = 0. \]

Hence \( k = \text{constant} \), unless \( \vec{N} \cdot X = 0 \).

Let \( R \) be the set of points of \( S \) at which \( \vec{N} \cdot X = 0 \). Then every point of \( R \) (if there is any) is not an interior point; for otherwise, \( S \) would contain a piece of cone with vertex 0. Hence every point of \( R \) is a limiting point of \( S - R \), and, due to the continuity of \( k = H/\vec{H} \),

\[ k = \text{constant throughout} \ S. \]

Moreover \( \vec{N} = \vec{N} \) implies that \( k \) is positive.

Consequently \( h \) is a homothetic transformation with center 0 and with positive constant of proportionality.

Case 2. \( 0 \in S \) or \( 0 \not\in S \). Without loss of generality, we may assume \( 0 \in S \). In any open set \( U \) of \( S \) containing 0, take a neighborhood \( V \) of 0. Let \( V' \) be the boundary of \( V \) (and so also of \( S - V \)). Since \( (1 - \vec{N} \cdot \vec{N}) \)

\[ dA \text{ always keeps the same sign} \]

(2.3) \[ \left| \int \int_{s-U} (1 - \vec{N} \cdot \vec{N})dA \right| \leq \left| \int \int_{s-V} (1 - \vec{N} \cdot \vec{N})dA \right|. \]

The expression on the right of (2.3) is equal to

\[ \frac{1}{2} \left| \int_{V'} [(\vec{N} - \vec{N}) \times \vec{X}] \cdot dX \right| \]

because of (2.2) and Stokes' Theorem, and it can be made as small as we please by choosing \( V \) small enough, while the expression on the left of (2.3) remains fixed. Hence

\[ \int \int_{s-U} (1 - \vec{N} \cdot \vec{N})dA = 0. \]
Following the same argument as in Case 1, we have \( k = \) positive constant in \( S - U \) for every open set \( U \) of \( S \) containing 0. Hence \( k = \) positive constant throughout \( S \), since \( k \) is continuous.

Assume \( \overline{H}X = -HX \). Write \( X = -kX \) where \( k = \frac{H}{\overline{H}} \). Through similar arguments as above, we obtain

\[
\int \int_{S} (1 + \overline{N} \cdot N) dA = 0,
\]

which gives \( \overline{N} = -N \) and therefore \( k = \) positive constant.

**Remark 1.** Theorem 1 still holds when \( S \) and \( \overline{S} \) are not closed but bounded with boundaries \( B \) and \( \overline{B} \), such that \( h(B) = \overline{B} \) and at corresponding points on \( B \) and \( \overline{B} \), we have \( \overline{N} = N \) for case (i) or \( \overline{N} = -N \) for case (ii). This is evident, because

\[
\int \int_{S} (1 - \overline{N} \cdot N) dA = \frac{1}{2} \int_{B} \left[ (N - \overline{N}) \times X \right] \cdot dX \quad \text{for case (i)},
\]

\[
\int \int_{S} (1 + \overline{N} \cdot N) dA = \frac{1}{2} \int_{B} \left[ (\overline{N} + N) \times X \right] \cdot dX \quad \text{for case (ii)}.
\]

**Remark 2.** If we consider the more general condition \( HX = r\overline{H}X \), where \( r \) is a constant, without loss of generality, we may assume \( |r| \leq 1 \). Then instead of (2.2) we get

\[
\frac{1}{2} d(a - rb) = (1 - rN \cdot \overline{N}) dA.
\]

Hence

\[
\int \int_{S} (1 - r\overline{N} \cdot N) dA = 0.
\]

This equation implies

\[
1 - rN \cdot \overline{N} = 0
\]

which is impossible unless \( r = \pm 1 \).

**Corollary.** Given a closed orientable surface \( S \) convex with respect to a fixed point 0. With 0 as origin, the quantities \( X, H, X', H' \) at points corresponding under \( f \) are related to each other by \( H'X' = -HX \). Then \( S \) is symmetric with respect to 0.

**Proof.** It is clear that \( f \) satisfies the assumptions in Theorem 1. Hence it is a homothetic transformation with center 0 and with negative constant of proportionality \( k \). Since both \( PP' \) and \( P'(P)' \)
pass through 0, \((P')'\) should be either \(P\) or \(P'\), and since \(f\) is one-one, \((P')' = P\). Hence \(k^2 = 1\), and \(k = 1\). Therefore \(S\) is symmetric with respect to 0.

**Proof of Theorem 2.** There is a transformation \(g\) in \(E^3\) (which is either a single rotation about an axis through 0 or such a rotation followed by a reflection against a plane through 0), such that each straight line \(0P\) is transformed into \(0P'\) where \(P, P'\) are points corresponding under \(h\).

Let \(S^* = g(S)\). It is clear that \(h^{-1}: S^* \rightarrow \overline{S}\) satisfies the assumptions of Theorem 1, and hence is a homothetic transformation with center 0. Therefore \(h\) is a similarity.

**Proof of Theorem 3.** Let \(g\) be the inversion about the unit sphere with center 0. Denote by \(X^*, H^*\) and \(N^*\) the position vector, the mean curvature and the unit vector along inward normal direction at \(P^* = g(P)\) of \(S^* = g(S)\), respectively. By simple calculations we obtain

\[
H^*X^* = -\left(H + 2\frac{X \cdot N}{X \cdot X}\right)X,
\]

which reduces to \(H^*X^* = \overline{H}X\) for case (i) and to \(H^*X^* = -HX\) for case (ii). Hence \(h^{-1}: S^* \rightarrow \overline{S}\) is a homothetic transformation with center 0 and with positive or negative constant of proportionality according as (i) or (ii) holds. Thus \(h = (h^{-1})g\) is an inversion about 0, and the radius of inversion is real or pure imaginary according as (i) or (ii) holds.

**Remark.** Theorem 3 still holds when \(S\) and \(\overline{S}\) are not closed but bounded with boundaries \(B\) and \(\overline{B}\) such that \(h(B) = \overline{B}\) and at corresponding points on the boundaries \(\overline{N} = -N + 2(X \cdot N/X \cdot X)X\) for case (i) or \(\overline{N} = N - 2(X \cdot N/X \cdot X)X\) for case (ii). This is evident because \(N^* = -N + 2(X \cdot N/X \cdot X)X\).

**Corollary.** If \(S\) is a closed orientable surface convex with respect to a point 0 not on \(S\), and with 0 as origin we have \(H = -(X \cdot N/X \cdot X)X\). Then \(S\) is a sphere with center 0.

**Proof.** Since \(HX = -(H + 2(X \cdot N/X \cdot X))X\), each point of \(S\) is invariant under the inversion about a sphere with center 0 and with real radius. Consequently, \(S\) itself is a sphere with center 0.

**Reference**


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