1. **Introduction.** The purpose of this paper is to study some elementary transformations of surfaces embedded in a 3-dimensional Euclidean space $E^3$. This will be developed analogously to the following theorem [1]:

**Translation Theorem.** Given two closed orientable surfaces $S$, $\overline{S}$ and a homeomorphism $h: S \rightarrow \overline{S}$ such that: (1) each line joining corresponding points is parallel to a fixed direction $E$, (2) the mean curvatures at corresponding points are equal; moreover $S$, $\overline{S}$ are assumed not to contain pieces of cylinders in $E$-direction. Then $h$ is a translation.

All surfaces mentioned will be of class $C^2$. The notations in [1] will be adopted except that German letters will be replaced by corresponding capital English ones. For example: $X$, $N$, $H$ and $dA$ are respectively used to denote the position vector, the unit vector along inward normal direction, the mean curvature and the surface element of a surface $S$. When a second surface $\overline{S}$ is mentioned, the corresponding quantities are represented by the same letters with bars above them. As in [1], the following formulas will be used:

\begin{align}
(1.1) & \quad dX \times dX = 2NdA, \\
(1.2) & \quad dX \times dN = -2HNdA.
\end{align}

A closed nonself-intersecting surface $S$ is said to be convex with respect to a given point 0, (1) if every straight line through 0 meets $S$ at no point, at one point of contact or at two distinct points, (2) if there is a differentiable homeomorphism $f: S \rightarrow S$ such that each straight line joining corresponding points passes through 0.

We intend to prove the following theorems:

**Theorem 1.** Given two closed orientable surfaces $S$, $\overline{S}$ and a differentiable homeomorphism $h: S \rightarrow \overline{S}$ such that: (1) each straight line $PP$ joining the corresponding points $P$ and $P$ passes through a fixed point 0; (2) with 0 as origin, the quantities $X$, $\overline{X}$, $H$, $\overline{H}$ are related to each other either by (i) $\overline{H}X = HX$ throughout $S$ and $\overline{S}$ or by (ii) $\overline{H}X = -HX$ throughout $S$ and $\overline{S}$. Moreover $S$, $\overline{S}$ are assumed not to contain pieces of cones with vertex 0. Then $h$ is a homothetic transformation with center 0 and with a

---

Received by the editors August 18, 1958.
positive or negative constant of proportionality according as (i) or (ii) holds.

**Theorem 2.** Given two closed orientable surfaces $S$, $S$ and a differentiable homeomorphism $h: S \to S$, such that: (1) each segment $PP$ joining the corresponding points $P$ and $P$ subtends a constant angle $POP$ about a fixed point $O$, (2) with $O$ as origin, $HX$ and $H\overline{X}$ are equal in magnitude. Moreover, $S$, $S$ are assumed not to contain pieces of cones with vertex $O$. Then $h$ is a similarity with $0$ as center of similitude.

**Theorem 3.** Given two closed orientable surfaces $S$, $S$ and a differentiable homeomorphism $h: S \to S$ such that: (1) each straight line $PP$ joining corresponding points $P$ and $P$ passes through a fixed point $O$; (2) with $O$ as origin, the quantities $X$, $\overline{X}$, $H$, $\overline{H}$ are related to each other either by (i) $H\overline{X} = -(H+2X \cdot N/X \cdot X)X$ throughout $S$ and $S$; or by (ii) $H\overline{X} = (H+2X \cdot N/X \cdot X)X$ throughout $S$ and $S$. Moreover $S$, $S$ are assumed not to contain either pieces of cones with vertex $0$ or the point $0$ itself. Then $h$ is an inversion with center $0$ and with real or pure imaginary radius of inversion according as (i) or (ii) holds.

I am grateful to Professor T. K. Pan for his encouragement during the preparation of this paper.

2. Proofs of theorems.

**Proof of Theorem 1.** Assume $H\overline{X} = HX$. Write $\overline{X} = kX$, where $k = H/\overline{H}$.

**Case 1.** $0 \not\in S$ and $0 \not\in \overline{S}$. Then $k \neq 0$, $\infty$.

$$d\overline{X} \times d\overline{X} = (kdX + Xdk) \times (kdX + Xdk)$$

$$= k^2(dX \times dX) + 2k(Xdk \times dX).$$

By (1.1), we have

$$\overline{N}d\overline{A} = k^2NdA + k(Xdk \times dX),$$

whose scalar product with $H\overline{X}(=HX)$ gives

$$a = (N \times X) \cdot dX, \quad b = (\overline{N} \times X) \cdot dX$$

and note that $ddX = 0$, then by use of (1.1) and (1.2), we obtain

$$da = 2(X \cdot N)HdA + 2dA,$$

$$db = -X \cdot (d\overline{N} \times dX) + 2(\overline{N} \cdot N)dA$$

$$= (2/k^2)(\overline{X} \cdot \overline{N})Hd\overline{A} + 2(\overline{N} \cdot N)dA$$

since
\[ \overline{X} \cdot (d\overline{N} \times d\overline{X}) = k^2 X \cdot (d\overline{N} \times dX). \]

Hence, by (2.1) we have

\[ (2.2) \quad d(a - b)/2 = (1 - \overline{N} \cdot N)dA. \]

From Stokes' Theorem, it is evident that

\[ \int \int_S (1 - \overline{N} \cdot N)dA = 0. \]

Since \( 1 - \overline{N} \cdot N \geq 0 \), and \( dA \) always keeps the same sign, we have

\[ 1 - \overline{N} \cdot N = 0 \]

and therefore

\[ N = \overline{N}. \]

Moreover, since \( \overline{N} \cdot d\overline{X} = 0, N \cdot dX = 0 \), we have

\[ (N \cdot X)dk = 0. \]

Hence \( k = \text{constant} \), unless \( N \cdot X = 0 \).

Let \( R \) be the set of points of \( S \) at which \( N \cdot X = 0 \). Then every point of \( R \) (if there is any) is not an interior point; for otherwise, \( S \) would contain a piece of cone with vertex 0. Hence every point of \( R \) is a limiting point of \( S - R \), and, due to the continuity of \( k = H/\overline{H} \), \( k = \text{constant} \) throughout \( S \). Moreover \( N = \overline{N} \) implies that \( k \) is positive. Consequently \( h \) is a homothetic transformation with center 0 and with positive constant of proportionality.

**Case 2.** \( \partial S \) or \( \partial \overline{S} \). Without loss of generality, we may assume \( 0 \in S \). In any open set \( U \) of \( S \) containing 0, take a neighborhood \( V \) of 0. Let \( V' \) be the boundary of \( V \) (and so also of \( S - V \)). Since \( (1 - \overline{N} \cdot N) \) \( dA \) always keeps the same sign

\[ (2.3) \quad \left| \int \int_{s - u} (1 - \overline{N} \cdot N)dA \right| \leq \left| \int \int_{s - v} (1 - \overline{N} \cdot N)dA \right|. \]

The expression on the right of (2.3) is equal to

\[ \frac{1}{2} \left| \int_{V'} [(N - \overline{N}) \times X] \cdot dX \right| \]

because of (2.2) and Stokes' Theorem, and it can be made as small as we please by choosing \( V \) small enough, while the expression on the left of (2.3) remains fixed. Hence

\[ \int \int_{s - u} (1 - \overline{N} \cdot N)dA = 0. \]
Following the same argument as in Case 1, we have $k = \text{positive constant}$ in $S - U$ for every open set $U$ of $S$ containing $0$. Hence $k = \text{positive constant}$ throughout $S$, since $k$ is continuous.

Assume $\overline{H} \overline{X} = -H X$. Write $\overline{X} = -k X$ where $k = H / \overline{H}$. Through similar arguments as above, we obtain

$$\int \int_{S} (1 + \overline{N} \cdot N) dA = 0,$$

which gives $\overline{N} = -N$ and therefore $k = \text{positive constant}$.

Remark 1. Theorem 1 still holds when $S$ and $\overline{S}$ are not closed but bounded with boundaries $B$ and $\overline{B}$, such that $h(B) = \overline{B}$ and at corresponding points on $B$ and $\overline{B}$, we have $\overline{N} = N$ for case (i) or $\overline{N} = -N$ for case (ii). This is evident, because

$$\int \int_{S} (1 - \overline{N} \cdot N) dA = \frac{1}{2} \int_{B} [(N - \overline{N}) \times X] \cdot dX \quad \text{for case (i)},$$

$$\int \int_{S} (1 + \overline{N} \cdot N) dA = \frac{1}{2} \int_{B} [(\overline{N} + N) \times X] \cdot dX \quad \text{for case (ii)}.$$

Remark 2. If we consider the more general condition $H X = r \overline{H} \overline{X}$, where $r$ is a constant, without loss of generality, we may assume $|r| \leq 1$. Then instead of (2.2) we get

$$\frac{1}{2} d(a - rb) = (1 - r N \cdot \overline{N}) dA.$$

Hence

$$\int \int_{S} (1 - r \overline{N} \cdot N) dA = 0.$$

This equation implies

$$1 - r N \cdot \overline{N} = 0$$

which is impossible unless $r = \pm 1$.

Corollary. Given a closed orientable surface $S$ convex with respect to a fixed point $0$. With $0$ as origin, the quantities $X$, $H$, $X'$, $H'$ at points corresponding under $f$ are related to each other by $H' X' = -H X$. Then $S$ is symmetric with respect to $0$.

Proof. It is clear that $f : S \rightarrow S$ satisfies the assumptions in Theorem 1. Hence it is a homothetic transformation with center $0$ and with negative constant of proportionality $-k$. Since both $P P'$ and $P'(P')'$
pass through 0, \((P')'\) should be either \(P\) or \(P'\), and since \(f\) is one-one, \((P')'=P\). Hence \(k^2=1\), and \(k=1\). Therefore \(S\) is symmetric with respect to 0.

**Proof of Theorem 2.** There is a transformation \(g\) in \(E^3\) (which is either a single rotation about an axis through 0 or such a rotation followed by a reflection against a plane through 0), such that each straight line \(0P\) is transformed into \(0\overline{P}\) where \(P\), \(\overline{P}\) are points corresponding under \(g\).

Let \(S^*=g(S)\). It is clear that \(h^{-1}: S^*\to \overline{S}\) satisfies the assumptions of Theorem 1, and hence is a homothetic transformation with center 0. Therefore \(h\) is a similarity.

**Proof of Theorem 3.** Let \(g\) be the inversion about the unit sphere with center 0. Denote by \(X^*, H^*\) and \(N^*\) the position vector, the mean curvature and the unit vector along inward normal direction at \(P^*=g(P)\) of \(S^*=g(S)\), respectively. By simple calculations we obtain

\[
H^*X^* = -\left(H + 2\frac{X \cdot N}{X \cdot X}\right)X,
\]

which reduces to \(H^*X^* = \overline{H}X\) for case (i) and to \(H^*X^* = -HX\) for case (ii). Hence \(h^{-1}: S^*\to \overline{S}\) is a homothetic transformation with center 0 and with positive or negative constant of proportionality according as (i) or (ii) holds. Thus \(h = (h^{-1})g\) is an inversion about 0, and the radius of inversion is real or pure imaginary according as (i) or (ii) holds.

**Remark.** Theorem 3 still holds when \(S\) and \(\overline{S}\) are not closed but bounded with boundaries \(B\) and \(\overline{B}\) such that \(h(B) = \overline{B}\) and at corresponding points on the boundaries \(\overline{N} = -N - 2(X \cdot N/X \cdot X)X\) for case (i) or \(\overline{N} = N - 2(X \cdot N/X \cdot X)X\) for case (ii). This is evident because \(N^* = -N + 2(X \cdot N/X \cdot X)X\).

**Corollary.** If \(S\) is a closed orientable surface convex with respect to a point 0 not on \(S\), and with 0 as origin we have \(H = -(X \cdot N/X \cdot X)X\). Then \(S\) is a sphere with center 0.

**Proof.** Since \(HX = -(H + 2(X \cdot N/X \cdot X))X\), each point of \(S\) is invariant under the inversion about a sphere with center 0 and with real radius. Consequently, \(S\) itself is a sphere with center 0.

**Reference**


**National Taiwan University, Taiwan, China**