# THE BOHR SPECTRUM OF A FUNCTION

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C. S. Herz, [1], conjectured that

$$\lim_{N\to\infty} \frac{1}{2N}\int_{-N}^{N} \exp(-itx)\phi(x)dx = 0 \tag{1}$$

for almost all real values of $t$ if $\phi(x)$ is bounded and uniformly continuous. H. G. Eggleston, [2], has proved the truth of this conjecture and shown that the exceptional set of values of $t$ may be nondenumerable.

The main object of this note is to point out that Herz's result follows readily from a theorem of Zygmund [3] even when the hypotheses are considerably relaxed. The restriction of uniform continuity may be omitted, and that of boundedness replaced by, for instance,

$$\phi(x) = O(|x|^q)$$

for large $|x|$, where $q<1/2$. This is a consequence of Theorem 2.

It may be noted also, although this does not imply Herz's theorem, that (1) is true for all real values of $t$ other than zero if the two functions $\phi(\pm x)/x$ are of bounded variation in $(c, \infty)$ for some $c>0$ and tend to zero as $x\to\infty$. This is a consequence of Theorem 1.

It is assumed that the functions $f$ in Lemma 1 and $\phi$ in Theorems 1 and 2 are locally integrable in the sense of Lebesgue.

LEMMA 1. *If*

$$\lim_{N\to\infty}\int_1^N x^{-1}f(x)dx$$

*exists, then*

$$\lim_{N\to\infty} N^{-1}\int_1^N f(x)dx = 0.$$

LEMMA 2. *If $g\in L^p(a, \infty)$ for some $p$ such that $1\leqq p<2$, then the integral*

$$\int_a^\infty \exp(-itx)g(x)dx$$

*is convergent for almost all real values of $t$.*

THEOREM 1. *If the two integrals*

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$$\int_1^\infty \exp(\mp itx) \frac{\phi(\pm x)}{x} dx$$

*are convergent for a particular value of $t$, then* (1) *holds.*

THEOREM 2. If $\phi(\pm x)/x \in L^p(a, \infty)$ *for some $a>0$ and some $p$ such that $1 \leqq p < 2$, then* (1) *holds for almost all real values of $t$.*

The proof of Lemma 1 is simple and is omitted. The lemma is an analogue for integrals of a particular case of a theorem on sums due to Kronecker.[1]

Lemma 2 is virtually the theorem of Zygmund [**3**] of which a proof has also been given by Kac [**4**].

In the proof of Theorem 1 it may be assumed that $\phi(x)=0$ for $|x|<1$. Then

$$\frac{1}{2N}\int_{-N}^N \exp(-itx)\phi(x)dx = \frac{1}{2N}\int_1^N f(x)dx,$$

where $f(x)=\exp(-itx)\phi(x)+\exp(itx)\phi(-x)$. The conclusion of Theorem 1 now follows from Lemma 1.

The hypothesis of Theorem 2 implies, by Lemma 2 with $g(x) =\phi(\pm x)/x$, that for almost all real values of $t$ the hypothesis of Theorem 1 is satisfied. Theorem 2 therefore follows from Theorem 1.

## REFERENCES

**1.** C. S. Herz, *The Bohr spectrum of bounded functions*, Bull. Amer. Math. Soc. vol. 62 (1955) p. 76.

**2.** H. G. Eggleston, *The Bohr spectrum of a bounded function*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 328–332.

**3.** A. Zygmund, *A remark on Fourier transforms*, Proc. Cambridge Philos. Soc. vol. 32 (1936) pp. 321–327.

**4.** M. Kac, *On a theorem of Zygmund*, Proc. Cambridge Philos. Soc. vol. 47 (1951) pp. 475–476.

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[1] For Kronecker's theorem see G. H. Hardy, *Divergent series*, Oxford, 1949, p. 73, Theorem 26 and p. 91; or K. Knopp, *Theory and application of infinite series*, London and Glasgow, 1928, p. 129, Theorem 3. The second paper attributed to Kronecker by Hardy on p. 91 of the first reference is in fact by J. L. W. V. Jensen. The error occurs also in the book of Pringsheim cited by Hardy.