THE BOHR SPECTRUM OF A FUNCTION

J. COSSAR

C. S. Herz, [1], conjectured that

\[ \lim_{N \to \infty} \frac{1}{2N} \int_{-N}^{N} \exp \left( -ix \phi(x) \right) dx = 0 \]

for almost all real values of \( t \) if \( \phi(x) \) is bounded and uniformly continuous. H. G. Eggleston, [2], has proved the truth of this conjecture and shown that the exceptional set of values of \( t \) may be non-denumerable.

The main object of this note is to point out that Herz's result follows readily from a theorem of Zygmund [3] even when the hypotheses are considerably relaxed. The restriction of uniform continuity may be omitted, and that of boundedness replaced by, for instance,

\[ \phi(x) = O(|x|^g) \]

for large \( |x| \), where \( g < 1/2 \). This is a consequence of Theorem 2.

It may be noted also, although this does not imply Herz's theorem, that (1) is true for all real values of \( t \) other than zero if the two functions \( \phi(\pm x)/x \) are of bounded variation in \((c, \infty)\) for some \( c > 0 \) and tend to zero as \( x \to \infty \). This is a consequence of Theorem 1.

It is assumed that the functions \( f \) in Lemma 1 and \( \phi \) in Theorems 1 and 2 are locally integrable in the sense of Lebesgue.

**Lemma 1.** If

\[ \lim_{N \to \infty} \int_{1}^{N} x^{-1} f(x) dx \]

exists, then

\[ \lim_{N \to \infty} N^{-1} \int_{1}^{N} f(x) dx = 0. \]

**Lemma 2.** If \( g \in L^p(a, \infty) \) for some \( p \) such that \( 1 \leq p < 2 \), then the integral

\[ \int_{a}^{\infty} \exp (-ix) g(x) dx \]

is convergent for almost all real values of \( t \).

**Theorem 1.** If the two integrals

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are convergent for a particular value of \( t \), then (1) holds.

**Theorem 2.** If \( \phi(\pm x)/x \in L^p(a, \infty) \) for some \( a > 0 \) and some \( p \) such that \( 1 \leq p < 2 \), then (1) holds for almost all real values of \( t \).

The proof of Lemma 1 is simple and is omitted. The lemma is an analogue for integrals of a particular case of a theorem on sums due to Kronecker.\(^1\)

Lemma 2 is virtually the theorem of Zygmund [3] of which a proof has also been given by Kac [4].

In the proof of Theorem 1 it may be assumed that \( \phi(x) = 0 \) for \( |x| < 1 \). Then

\[
\frac{1}{2N} \int_{-N}^{N} \exp(-itx)\phi(x)dx = \frac{1}{2N} \int_{1}^{N} f(x)dx,
\]

where \( f(x) = \exp(-itx)\phi(x) + \exp(itx)\phi(-x) \). The conclusion of Theorem 1 now follows from Lemma 1.

The hypothesis of Theorem 2 implies, by Lemma 2 with \( g(x) = \phi(\pm x)/x \), that for almost all real values of \( t \) the hypothesis of Theorem 1 is satisfied. Theorem 2 therefore follows from Theorem 1.

**References**
