Bibliography


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QUASI-NIL RINGS

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Rings have been studied which have among others the property that every commutator $xy - yx$ is in the nucleus. It seems appropriate to consider rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and characteristic different from 2 it can be shown that such rings are either associative or have the property that $x^2 = 0$, for every element $x$ of the ring. If further $(x, y, z) + (y, z, x) + (z, x, y)$ is in the nucleus for all elements $x, y, z$ of the ring, then the ring is either associative or a Lie ring.

We use the notation $(x, y, z) = (xy)z - x(yz)$. The nucleus $N$ of a ring $R$ consists of all $n \in R$ such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. $N$ is a subring of $R$.

Lemma. Let $R$ be a prime ring satisfying $x^2 \in N$ for every $x \in R$ and of characteristic different from 2. Then either $R$ is associative or $N^2 = 0$.

Proof. For all $r, s \in R$, $rs + sr = (r + s)^2 - r^2 - s^2$ must be in $N$. Select $n, n' \in N$, and $x, y, z \in R$. Then $(n(n'x + xn'), y, z) = 0$, so that $(nn'x, y, z) = -(nxn', y, z)$. Similarly $(nn'x, y, z) = -(nxn', y, z)$ and $(xn', y, z) = -(nn'x, y, z)$. By combining these three equalities it follows that $2(nn'x, y, z) = 0$. Assuming characteristic not 2 it then follows that $(nn'x, y, z) = 0$. Since $(nx, y, z) = (nx)y - (nx)(yz) = (n(xy))z - n((xy)z) = n((xy)z - n(x(yz)) = n(x, y, z)$, we replace $n$
by \( nn' \) and obtain \((nn'x, y, z) = nn'(x, y, z) \). We deduce that \( nn'(x, y, z) = 0 \), so that \( N^2(R, R, R) = 0 \). Let \( I \) be the ideal generated by \( N^2 \), and \( J \) the ideal generated by all associators \((R, R, R)\). We note that 
\[
nn'x = n(n'x + xn') -(nx+xn)n'+xnn',
\]
so that \( N^2R \subseteq RN^2 + N^2 \). Consequently \( I = RN^2 + N^2 \). In an arbitrary ring \( J = (R, R, R) + (R, R, R)R \). It follows from \( N^2(R, R, R) = 0 \), that \( IJ = 0 \). Since \( R \) is prime either \( I = 0 \), or \( J = 0 \). If \( I = 0 \), then \( N^2 = 0 \). On the other hand if \( J = 0 \), then \( R \) is associative. This completes the proof of the lemma.

Theorem 1. Let \( R \) be a prime ring of characteristic not 2 satisfying 
\( r^2 \subseteq N \) for all \( r \in R \). Then either \( R \) is associative or \( r^2 = 0 \), for all \( r \in R \).

Proof. Let us consider the case \( N \neq R \). Then it follows from the lemma that \( N^2 = 0 \). Let \( K = N + NR \). Since \( rn = (rn + nr) - nr \in K \) and 
\[
\begin{align*}
snr &= (sn + ns)r - nsr \in K, 
\end{align*}
\]
for all \( n \in N \) and \( r, s \in R \), \( K \) must be an ideal of \( R \). Moreover if \( n' \in N \), then 
\[
nnn' = n(rn' + n'r) - nn'r = 0,
\]
since \( N^2 = 0 \). This suffices to show \( K^2 = 0 \). But \( R \) is prime and so \( K = 0 \). But then \( N = 0 \), whence \( r^2 = 0 \), for all \( r \in R \). This completes the proof of the theorem.

Theorem 2. Let \( R \) be a prime ring of characteristic not 2 satisfying 
(i) \( x^2 \subseteq N \) for all \( x \in R \), and (ii) \((x, y, z) + (y, z, x) + (z, x, y) \subseteq N \) for all \( x, y, z \in R \). Then \( R \) is either associative or a Lie ring. Conversely all associative rings and all Lie rings satisfy both (i) and (ii).

Proof. Assume that \( R \) satisfies (i) and (ii), and suppose \( N \neq R \). Then Theorem 1 implies that \( x^2 = 0 \), for all \( x \in R \), and consequently \( R \) is anti-commutative. For any \( n \in N \), \( nxy = -nxy = xyn \), and also \( nxy = -xyn \), so that \( 2nxy = 0 \). Thus \( NR^2 = 0 \). The set \( T \) of all \( t \in R \) such that \( tR = 0 \), forms an ideal of \( R \) which must be zero since \( R \) is prime. But since \( NR \subseteq T \) we obtain first \( NR = 0 \), and subsequently \( N = 0 \). In any anti-commutative ring \((x, y, z) + (y, z, x) + (z, x, y) = (xy)z - x(yz) + (yz)x - y(zx) + (zx)y - z(xy) = 2((xy)z + (yz)x + (zx)y)\) which equals twice the Jacobian of \( x, y, z \). Then because of (ii) we conclude that the well known Jacobi identity holds and thus \( R \) is a Lie ring. The converse follows automatically. In an associative ring \( N = R \), so that (i) and (ii) are trivially satisfied. All Lie rings are anti-commutative and satisfy the Jacobi identity, so that (ii) follows. This completes the proof of the theorem.

N. H. McCoy [1] and R. L. SanSoucie [2] have shown independently that any primitive ring is prime. Consequently the above theorems may be extended to primitive and semi-simple rings in the usual way.

We conclude with a couple of examples.
Example 1. Let \( x, y, z, n \) be basis elements of the algebra \( A \) over an arbitrary field \( F \). All products of basis elements are defined to be zero with the exception of \( xy = -yx = z, \ xz = -xz = y, \ yz = -zy = x, \) and \( n^2 = n \). For every \( \alpha, \beta, \gamma, \delta \in F \), \( (\alpha x + \beta y + \gamma z + \delta n)^2 = \delta^2 n \) is in the nucleus. Thus for every \( r \in R \), \( r^2 \in N \). \( R \) has four ideals, the trivial ones, the ideal \( B \) generated by \( x, y, z \), and the ideal \( C \) generated by \( n \). Also \( BC = 0 = CB \), while \( B^2 = B \), and \( C^2 = C \).

Example 2. Let \( 1, x, y \) be basis elements of the algebra \( R \) over an arbitrary field \( F \), where \( xy = 1, \ yx = x^2 = y^2 = 0 \). For any \( \alpha, \beta, \gamma \in F \), \( (\alpha + \beta x + \gamma y)^2 = \alpha^2 + 2\alpha \beta x + 2\alpha \gamma y + \beta \gamma = 2\alpha (\alpha + \beta x + \gamma y) + \beta \gamma - \alpha^2 \). Thus \( R \) is quadratic over \( F \). Moreover it can be readily verified that \( R \) is simple, power-associative, and that all commutators of \( R \) are contained in \( F \). \( R \) is not associative since \( (x, y, y) = y \). Also \( (x + y)^2 = 1 \neq 0 \). If \( F \) happens to be a field of characteristic 2 then \( r^2 \in F \) for every \( r \in R \). We see that Theorem 1 fails to hold for rings of characteristic 2.

Bibliography


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