THE ANALYSIS OF THE CHARACTERS OF THE LIE REPRESENTATIONS OF THE GENERAL LINEAR GROUP

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1. If a free Lie ring has \( n \) generators and \( A \) is a nonsingular \( n \times n \) matrix of complex elements, then when the generators undergo a linear transformation of matrix \( A \), the module of all forms of degree \( m \) in the generators is mapped into itself by a linear transformation of matrix \( L_m(A) \) on a set of basis elements. The mapping \( A \rightarrow L_m(A) \) is a representation of the full linear group known as the \( m \)th Lie representation \([1]\). The character of this representation has been shown \([2]\) to be \( \gamma_m = m^{-1} \sum_{d | m} \mu(d) s_d^{m/d} \), where \( \mu(k) \) is the Möbius function of the integer \( k \), and \( s_r \) is the sum of the \( r \)th powers of the eigenvalues of \( A \). The decomposition of the \( m \)th Lie representation into its irreducible constituents is in exact correspondence with the analysis of the symmetric function \( \gamma_m \) into Schur functions. When \( m \) is prime there is a simple rule for the coefficient \( \alpha_\lambda \) of any \( S \)-function \( \{ \lambda \} \) in \( \gamma_m \), \([2]\), but there is no such rule when \( m \) is composite. Since \( s_d^{m/d} = \sum_\lambda \chi_\lambda^{d|m} \{ \lambda \} \), where \( \chi_\lambda^{d|m} \) is the irreducible character of the class \( \rho \) of the symmetric group \( S_m \) corresponding to the partition \( (\lambda) \) of \( m \), it follows that \([10]\),

\[
\alpha_\lambda = \frac{1}{m} \sum_{d | m} \mu(d) \chi_\lambda^{d|m/d}.
\]

The calculation of \( \alpha_\lambda \) thus reduces to the calculation of characters of the form \( \chi_\lambda^{d|m/d} \).

It is the purpose of this note to suggest, in §2, a method of calculation of these characters, and, in §3, to indicate some relations satisfied by the \( \alpha_\lambda \).

2. The well known formulae for \( \chi_{1^m}^\lambda \) are

\[
\chi_{1^m}^\lambda = \frac{m! \prod_{r<s} (\lambda_r - \lambda_s - r + s)}{\prod_r (\lambda_r + \rho - r)!},
\]

where \( \rho = \) number of parts in \( (\lambda) \);

\[
\chi_{1^m}^\lambda = \frac{m!}{\prod_{r,s} (\lambda_r + \lambda_s - r - s + 1)}.
\]

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where \((\lambda) = (\lambda_1, \lambda_2, \cdots)\) is the partition conjugate to \((\lambda)\);

\[
\lambda \chi_{1^n} = \frac{m!}{H_{\lambda}},
\]

where \(H_{\lambda}\) is the product of the hook-lengths \(h_{ij}\) in the hook-graph of 
\((\lambda)\) \[3\]. These give the coefficient of \(\{\lambda\}\) in \(s_{1^n}^m\). An alternative 
method, which extends to \(\chi_{r^m/r}\), the coefficient of \(\{\lambda\}\) in \(s_{r^m/r}^r\), is by 
evaluation of certain determinants; \[4, pp. 134–135; 5\]. Thus to find 
\(\chi_{1^n}^{531^2}\) we evaluate

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
5! & 6! & 7! & 8! \\
1 & 1 & 1 & 1 \\
2! & 3! & 4! & 5! \\
1 & 1 & 1 & 1 \\
1! & 2! \\
\end{vmatrix}
\]

obtained respectively from

\[
\{531^2\} = \begin{vmatrix}
\{5\} & \{6\} & \{7\} & \{8\} \\
\{2\} & \{3\} & \{4\} & \{5\} \\
\{0\} & \{1\} & \{2\} \\
\{0\} & \{1\} \\
\end{vmatrix}
\text{ and } \{531^2\} = \begin{vmatrix}
\{51^3\} & \{21^3\} \\
\{5\} & \{2\} \\
\end{vmatrix},
\]

\[6\] giving 567. To find the coefficient of \(\{531^2\}\) in \(s_{10/r}^{10/r}\) we replace every 
\(\{k\}\) in the first of these determinants by zero if \(k\) is not a multiple 
of \(r\), and by \(1/(k/r)!\) if it is. Multiplying the determinant by \((10/r)!\) 
gives the required coefficient. Thus the coefficients of \(\{531^2\}\) in 
\(s_2^5, s_2^5, s_{10}\) are respectively

\[
\begin{vmatrix}
1 & 1 & 1 \\
3! & 4! & 5! \\
1 & 1 & 2! \\
\end{vmatrix}
= 15, \quad 2! \begin{vmatrix}
1 & 1 & 1 \\
1 & 2! & 3! \\
\end{vmatrix}
= 2, \quad \text{and zero,}
\]

giving the coefficient of \(\{531^2\}\) in \(\gamma_{10}\) as 55. This was given as 53 by 
Thrall \[1\], but was later corrected by Brandt \[2\].
Justification of the above procedure is given conveniently by the use of the differential operator \( D_\lambda \) obtained from \( \{ \lambda \} \) by replacing \( s^a \) by \( i^a \partial^a / \partial s^a \), [7]. Thus the coefficient of \( \{ \lambda \} \) in \( s^m \) is \( D_\lambda s^m \), and if \((\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_p)\), the coefficient can be written as

\[
\begin{vmatrix}
D_{\lambda_1} & D_{\lambda_1+1} & \cdots & D_{\lambda_1+p-1} \\
D_{\lambda_2-1} & D_{\lambda_2} & \cdots & D_{\lambda_2+p-2} \\
D_{\lambda_3-2} & D_{\lambda_3-1} & \cdots & D_{\lambda_3+p-3} \\
\vdots & \vdots & \ddots & \vdots \\
D_{\lambda_p-p+1} & \cdots & & D_{\lambda_p}
\end{vmatrix}
\]

and the only effective part of each \( D_k \) is \((k!)^{-1} \partial^{k}/\partial s^k\). Then in \( D_\lambda s^m/r \) the only effective part of each \( D_k \) is

\[
\frac{1}{k!} \frac{k!}{(k/r)!} \frac{\partial^{k/r}}{\partial s^{k/r}} = \frac{1}{(k/r)!} \frac{\partial^{k/r}}{\partial s^{k/r}}
\]

so that we get zero elements in the determinant when \( k \) is not a multiple of \( r \), and elements \(1/(k/r)! \) when \( k \) is a multiple of \( r \).

3. We now list a number of results concerning the \( \alpha_\lambda \).
I. When \( m \) is prime, \( \alpha_\lambda \) is the integer nearest to \( m^{-1} \chi^m_\lambda \) [2].
II. \( \alpha_m = 0 \) for \( m > 1 \), [10], and \( \alpha_m = 0 \) for \( m > 2 \).
These follow since \( \alpha_m = m^{-1} \sum \mu(d) \), and \( \alpha_1^m = m^{-1} \sum (-1)^{m+m/d} \mu(d) \).
III. \( \alpha_{m-1,1} = 1 \) for \( m > 1 \), [10], and \( \alpha_{21}^{m-2} = 1 \) for \( m > 2 \).
These follow since

\[
\chi^{m-1,1}_\lambda = m - 1, \quad \text{and} \quad \chi^{m-1,1}_\alpha = -1 \quad \text{for} \ a > 1, \ b \geq 1,
\]

[8, p. 137] and so

\[
\alpha_{m-1,1} = m^{-1} \left[ m - 1 - \sum_{d \neq 1} \mu(d) \right] = 1 \quad \text{for} \ m > 1,
\]

\[
\alpha_{21}^{m-2} = m^{-1} \left[ m - 1 - \sum_{d \neq 1} (-1)^{m+m/d} \mu(d) \right] = 1, \quad \text{for} \ m > 2.
\]

IV. If \( m \) is odd or a multiple of four, then \( \alpha_\lambda = \alpha_\lambda^\circ \). It is well known that \( \chi^{m/3}_d = \chi_{d^{m/3}}^\circ \) for odd values of \( d \), and also for even \( d \) whenever \( m/d \) is even. This proves IV.

When \( m \) is twice an odd integer, \( \chi^{m/3}_d = -\chi_{d^{m/3}}^\circ \) for even \( d \), and in this case \( \gamma_\lambda \), expressed in power sums, can be written as \( P + Q \), where the coefficients of \( \{ \lambda \} \) and \( \{ \lambda \}^\circ \) are the same in \( P \), but have opposite signs in \( Q \). \( P \) has all \( s^{m/3}_{d/3} \) with odd \( d \), and \( Q \) has all \( s^{m/3}_{d/3} \) with even \( d \). There are certain partitions \( (\lambda) \) for which the coefficient of \( \{ \lambda \} \) in \( Q \) is zero, and for these \( \alpha_\lambda = \alpha_\lambda^\circ \).
The following three results are typical of many which can be obtained by equating appropriate coefficients when the right hand side of

$$m^{-1} \sum_{d|m} \mu(d) s_d^{m/d} = \sum_{\lambda} \alpha_\lambda \{\lambda\}$$

is expressed in terms of power sums by writing

$$\{\lambda\} = (m!)^{-1} \sum_{\rho} h_\rho^\lambda S_\rho,$$

where $h_\rho$ is the order of the class ($\rho$) = $1^{s_1}2^{s_2}\cdots$ of $\mathcal{S}_m$, and $S_\rho = s_1^{s_1}s_2^{s_2}\cdots$.

V. $\sum_{\lambda} X_\lambda^m \alpha_\lambda = (m-1)!$. This is obtained by equating coefficients of $s_m^m$.

VI. $\sum_{\lambda=m-r,1r} (-1)^{\alpha_\lambda} = \mu(m)$. Obtained by equating coefficients of $s_m$.

VII. If $S$-functions of ranks one and two are written respectively in Frobenius notation as

$$\{\lambda\} = \left\{\begin{array}{c} X_r \\ Y_r \end{array}\right\} \quad \text{and} \quad \{\nu\} = \left\{\begin{array}{c} X_{t_1} X_{t_2} \\ Y_{t_1} Y_{t_2} \end{array}\right\},$$

and $a, b$ are any two unequal positive integers such that $a + b = m$, then $\sum_{\lambda} \theta_r \alpha_\lambda + \sum_{\nu} \phi_t \alpha_\nu = 0$, where

$$\theta_r = (-1)^{Y_r} \quad \text{if} \quad m > X_r \geq a,$$

$$= 0 \quad \text{if} \quad a > X_r \geq b,$$

$$= (-1)^{Y_r+1} \quad \text{if} \quad b > X_r \geq 0,$$

$$\phi_t = (-1)^{Y_{t_1}+Y_{t_2}} \quad \text{if} \quad X_{t_1} + Y_{t_1} = a - 1,$$

$$= (-1)^{Y_{t_1}+Y_{t_2}+1} \quad \text{if} \quad X_{t_1} + Y_{t_2} = a - 1,$$

$$= 0 \quad \text{otherwise.}$$

This result follows by equating coefficients of $s_a s_b$, [9]. Other special relations of this kind may be obtained by equating coefficients of $s_a s_b s_c$, and of other terms. They depend on a knowledge of expressions such as $\theta_r, \phi_t$ above for the appropriate characteristics of $\mathcal{S}_m$.

VIII. If $h_d$ is the order of the class ($d^{m/d}$) of $\mathcal{S}_m$, then

$$\sum_{\lambda} \alpha_\lambda^2 = \frac{(m-1)!}{m} \sum_d \frac{1}{h_d},$$

for square free values of $d$.

We have $D_{\gamma_m \gamma_m} = \sum_{\lambda} \alpha_\lambda D_{\lambda} \sum_{\lambda} \alpha_\lambda \{\lambda\} = \sum_{\lambda} \alpha_\lambda^2$. Also
\[ D_{\gamma_m \gamma_m} = m^{-1} \left[ \sum_d \mu(d) d^{m/d} \frac{\partial^{m/d}}{\partial s^{m/d}} \right] m^{-1} \sum_d \mu(d) s_d^{m/d} \]
\[ = (m^2)^{-1} \left[ \sum (\mu(d))^2 d^{m/d} (m/d)! \right] \]
\[ = (m^2)^{-1} \sum \frac{m!}{h_d} \]

for square free values of \( d \).

More explicit results may be written down when \( m \) has some prescribed form. Thus when \( m \) is a prime \( p \), \( \sum \alpha_x^2 = p^{-1}[1 + (p - 1)!] \), and when \( m \) is the product of two distinct primes \( p, q \),

\[ \sum \alpha_x^2 = \frac{1}{pq} [(pq - 1)! + p^{s-1}(q - 1)! + q^{s-1}(p - 1)! + 1]. \]

When \( m \) is not twice an odd number we have \( \alpha_x = \alpha_x^* \) by IV. But if \( m = 2(2k+1) \), then by evaluating \( D_{\gamma_m \gamma_m} \) in two ways, we obtain a further result;

IX. If \( m \) is twice an odd number, then \( \sum \alpha_x \alpha_x^* = (m-1)!/m \cdot \sum_d (-1)^{d+1} (h_d)^{-1} \) for square free values of \( d \).

Some results on the characters of the Lie representations in the special case when the number of generators is two have been given in a recent paper by Davis [10].

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References


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