COMMENT ON A PAPER OF C. ULUÇAY

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There is a gap in the proof given in [1] that the Bloch-Landau constant satisfies $\mathfrak{A} > .629$. After some correspondence with the author of [1] it appears that a satisfactory proof of the result is not available. Reproduced below are the steps which lead to the gap.

The function $f(z)$ is univalent and satisfies

$$|f(z)| \leqq \frac{1}{2} \log \frac{1+|z|}{1-|z|} = |z| M(|z|), \; |z| < 1.$$

For each fixed s, $0<s<1$, $f(z, s) = f(sz)/s$ is univalent and satisfies $|f(z, s)| \leqq M(s)$ for $|z|<1$. For $t>0$, $\Phi(z') = z'/(1 \pm tz')^2$ is univalent for $|z'| < 1/t$ and is not univalent (or regular) in any larger concentric disc. Also

$$\tilde{f}(z, s, t) = tM(s)\{\Phi[(f(z, s)/tM(s))^3]\}^{1/3}$$

is univalent for $|z|<1$ provided

$$|f(z, s)/tM(s)|^3 < 1/t, \qquad |z| < 1,$$

and the univalence of $\tilde{f}(z, s, t)$ for $|z|<1$ is thus assured only if $t>1$. If $f(z)$ omits c then $\tilde{f}(z, s, t)$ omits

$$\gamma(s) = (c/s)[1 \pm t^{-2}c^3/s^3M^3(s)]^{-2/3}.$$

If $|\arg c^3| \leqq \pi/2$ and the plus sign is used in $\gamma(s)$, then when $t=t_c$ is so chosen that $\gamma(s)>0$, the condition

$$\arg c^3 - 2 \arg [1 + c^3/t_c^2 s^3 M^3(s)] = 0$$

must hold. By an elementary theorem of geometry, $|c^3|/t_c^2 s^3 M^3(s) = 1$, and a similar argument shows that t_c satisfies this same relation in the case where $\pi/2 \leqq |\arg c^3| \leqq \pi$. The subsequent part of the proof in [1] involves the univalence of $\tilde{f}(z, s, t_c)$ as $s \to 1$. Because of the restriction on t_c only those values of s may be used for which $sM(s) < |c|$ and, since $M(s) \to \infty$ as $s \to 1$, it is not permissible to let $s \to 1$.

Added in proof. See review by E. Reich, Math. Rev. vol. 19 (1958) p. 736.

REFERENCE

1. C. Uluçay, *Bloch functions of the third kind and the constant* $\mathfrak{A}$, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 923–925.

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