PROXIMITY MAPS FOR CONVEX SETS
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The method of successive approximation is applied to the problem of obtaining points of minimum distance on two convex sets. Specifically, given a closed convex set \( K \) in Hilbert space, let \( P \) be the map which associates with each point \( x \) the point \( Px \) of \( K \) closest to \( x \). That \( P \) is well-defined is proved in [1, p. 6]. \( P \) will be called the proximity map for \( K \). If there are two such sets, \( K_1 \) and \( K_2 \), let \( Q \) denote the composition \( P_1P_2 \) of their proximity maps. It is shown that every fixed point of \( Q \) is a point of \( K_1 \) closest to \( K_2 \), and that the fixed points of \( Q \) may be obtained by iteration of \( Q \) when one of the sets is compact or when both are polytopes in \( E^n \). An application to the solution of linear inequalities is cited. Our thanks are due the referee for having suggested substantial simplifications.

**Theorem 1.** Let \( Q \) be a map of a metric space into itself such that
(i) \( d(Qx, Qy) \leq d(x, y) \),
(ii) if \( x \neq Qx \), then \( d(Qx, Q^2x) < d(x, Qx) \),
(iii) for each \( x \), the sequence \( Q^n x \) has a cluster point. Then for each \( x \), the sequence \( Q^n x \) converges to a fixed point of \( Q \).

**Proof.** By (i), the sequence \( d(Q^n x, Q^{n+1} x) \) is nonincreasing. Let \( y \) be a cluster point of \( Q^n x \), say \( y = \lim_k Q^n k x \). By (i), \( Q \) is continuous; therefore \( d(y, Qy) = \lim_k d(Q^n k x, Q^{n+k+1} x) = \lim_n d(Q^n x, Q^{n+1} x) = \lim_k d(Q^{n+k+1}, Q^{n+k+2} x) = d(Qy, Q^2y) \) contrary to (ii) unless \( y = Qy \). From (i) it follows that for all \( n \), \( d(Q^{n+N} x, y) \leq d(Q^N x, y) \) whence \( Q^n x \rightarrow y \).

**Corollary.** Let \( Q \) be a map of a normed linear space into itself having the property \( ||Qx - Qy|| \leq ||x - y|| \), equality holding only if \( x = y \). Let \( R = \alpha Q + (1 - \alpha) I \), \( (0 < \alpha \leq 1) \). If the range of \( R \) is compact, then \( Q \) has a unique fixed point which is the limit of every sequence \( R^n x \) with \( x \) arbitrary. (For related results, see [2].)

**Lemma.** Let \( K \) be a convex set in Hilbert space. A point \( b \in K \) is nearest a point \( a \in K \) if and only if \( s = (x - b, b - a) \geq 0 \) for all \( x \in K \).

**Proof.** Suppose \( b \) nearest \( a \), and let \( x \) be arbitrary in \( K \). When \( 0 \leq t \leq 1 \), \( tx + (1 - t)b \in K \). Thus \( 0 \leq ||a - tx - (1 - t)b||^2 - ||a - b||^2 = t^2 ||b - x||^2 + 2ts \). But this inequality would be violated by small \( t \)

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unless $s \geq 0$. For the converse, suppose $s \geq 0$. Then $\|x - a\|^2 - \|a - b\|^2 = (x, x) - 2(a, x) + 2(a, b) - (b, b) = (x - b, x - b) + 2s \geq 0$.

**Theorem 2.** Let $K_1$ and $K_2$ be two closed convex sets in Hilbert space. Let $P_i$ denote the proximity map for $K_i$. Any fixed point of $P_1P_2$ is a point of $K_1$ nearest $K_2$, and conversely.

**Proof.** Suppose $y = P_2x$ and $x = P_1y$. If $x = y$, the distance between $K_1$ and $K_2$ is thereby attained. Otherwise $x \notin K_2$ and $y \notin K_1$. If $u$ is arbitrary in $K_1$, then by the Lemma, $(u - x, x - y) \geq 0$ whence $(u, x - y) \geq (x, x - y)$. Similarly for arbitrary $v \in K_2$, $(v, y - x) \geq (y, y - x)$. Addition yields $(u - v, x - y) \geq (x - y, x - y)$ from which via the Schwartz inequality, $\|u - v\| \geq \|x - y\|$. For the converse, suppose that $\|x - P_2x\| \leq \|z - P_2z\|$ for all $z \in K_1$. Setting $z = P_1P_2x$ we have $\|z - P_2z\| \leq \|z - P_2x\| \leq \|x - P_2x\| \leq \|z - P_2z\|$ whence $z = x$ by the uniqueness of $z$.

**Theorem 3.** The proximity map $P$ for a closed convex set $K$ in Hilbert space satisfies the Lipschitz condition $\|Px - Py\| \leq \|x - y\|$, equality holding only if $\|x - Px\| = \|y - Py\|$. \[ \text{Proof.} \] By the Lemma, $A = (Px - Py, Py - y) \geq 0$ and $B = (Py - Px, Px - x) \geq 0$. Regrouping terms in the inequality $A + B \geq 0$ and using the Schwartz inequality, one has $\|Py - Px\| \cdot \|y - x\| \geq (Py - Px, Py - Px) = \|Py - Px\|^2$, whence $\|y - x\| \geq \|Py - Px\|$. Equality holds here only if $A = B = 0$ and if $Py - Px = \lambda(y - x)$. Thus $C = (y - x, Py - y) = 0$ and $D = (y - x, Px - x) = 0$. A computation shows then that $0 = A - B + C + D = \|Px - x\|^2 - \|Py - y\|^2$.

**Theorem 4.** Let $K_1$ and $K_2$ be two closed convex sets in Hilbert space and $Q$ the composition $P_1P_2$ of their proximity maps. Convergence of $Q^n x$ to a fixed point of $Q$ is assured when either (a) one set is compact, or (b) one set is finite dimensional and the distance between the sets is attained.

**Proof.** Theorem 3 implies that $Q$ satisfies (i) of Theorem 1. If $y = Qx \neq x \in K_1$, then $\|y - P_2y\| \leq \|y - P_2x\| < \|x - P_2x\|$. By Theorem 3, $\|Qx - Qy\| \leq \|P_2x - P_2y\| < \|x - y\|$. Thus $Q$ satisfies (ii) of Theorem 1. If the distance between the sets is attained, then $Q$ has a fixed point $y$ by Theorem 2, and $\|Q^n x\| \leq \|Q^n x - Qy\| + \|y\| \leq \|x - y\| + \|y\|$. Boundedness of $\|Q^n x\|$ and finite-dimensionality of $K_1$ suffice for (iii) of Theorem 1. $Q' = P_2P_1$ replaces $Q$ in these arguments when $K_2$ replaces $K_1$. But if $y$ is a fixed point of $Q'$, then $P_1y$ is a fixed point of $Q$.

**Theorem 5.** In a finite dimensional Euclidean space, the distance
between two polytopes is attained, a polytope being the intersection of a finite family of halfspaces.

Proof. First, the case when both polytopes are linear manifolds. Let $K_1$ be the linear span of $\{x_1, \cdots, x_m\}$ and $K_2$ the linear span of $\{y_1, \cdots, y_n\}$ translated by a vector $cy_0$. Assume the $x$'s and $y$'s each form orthonormal sets. A point $x = \sum \xi_i x_i$ is sought which minimizes $G = \|cy_0 + \sum (x, y_i) y_i - x\|^2$. $G$ is a positive definite quadratic function of $\xi_1, \cdots, \xi_m$, and therefore attains a minimum.

Now assume the validity of the theorem when one polytope is of dimension less than $n$ and the other is a linear manifold. (The validity for $n=1$ being established by the above.) Let $K_1$ be a polytope of dimension $n$ and $K_2$ a linear manifold. On each proper face of $K_1$ there is a point nearest $K_2$. There being only a finite number of faces, either one of these points is the required one or there is a point $x_0 \in K_1$ such that $d(x_0, K_2) < d(F, K_2)$ for all proper faces $F$ of $K_1$. In the latter case, let $y_0$ be chosen nearest to $K_2$ in the least linear manifold containing $K_1$. Since $d(x, K_2)$ is a convex function of $x$, the line segment $x_0y_0$ contains no point of any proper face of $K_1$. Therefore $y_0 \in K_1$ and must be the required point. The proof for the remaining case is as above, mutatis mutandis.

Linear inequalities. Consider the system of linear inequalities $(A^i, x) \leq b_i$ where $x \in E_n$ and $1 \leq i \leq m$. Let $K_1$ denote the range of the matrix $A$ and $K_2$ the orthant $\{y \in E_m: y_i \leq b_i \text{ all } i\}$. If $K_1$ and $K_2$ have a point in common, then any $x$ for which $Ax \in K_1 \cap K_2$ is a solution of the system. Even if the system is inconsistent, a point on $K_1$ closest to $K_2$ may be obtained by iteration of the map $Q = P_1 P_2$. These proximity maps are defined as follows. If $A$ is of rank $n$, $P_1 = A (A^T A)^{-1} A^T$. If $A$ is not necessarily of rank $n$, one may write $P_1 = B B^T$ where $B$ is a column-orthogonal matrix whose range is that of $A$. $P_2 y = z$ if and only if $z_i = y_i$ when $y_i \leq b_i$ and $z_i = b_i$ otherwise. Convergence of each sequence $Q^n x$ is guaranteed by Theorem 4.

**Bibliography**


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