ON PHI-FAMILIES

CHARLES E. WATTS

1. Introduction. The purpose of this note is to show that the notion of sections with support in a phi-family in the Cartan version of the Leray theory of sheaves can be avoided by the following expedient. One uses the phi-family to construct a new space in the manner of a one-point compactification. Then a sheaf on the original space is shown to yield a new sheaf on the new space whose cohomology (with unrestricted supports) is that of the original sheaf with restricted supports. A partial generalization to the Grothendieck cohomology theory is given.

2. Phi-families. A family $\mathcal{F}$ of subsets of a topological space $X$ is a family of supports [1] provided:

(I) each member of $\mathcal{F}$ is closed;
(II) if $F \in \mathcal{F}$, then each closed subset of $F$ is $\in \mathcal{F}$;
(III) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cup F_2 \in \mathcal{F}$.

Given a family of supports $\mathcal{F}$, we choose an object $\in \mathcal{X}$, and define $X' = \mathcal{U} \mathcal{F}$, $X^* = X' \cup \{\infty\}$. We then topologize $X^*$ by saying that a subset $U$ of $X^*$ is open iff either $U$ is open in $X'$ or else $X^* - U \in \mathcal{F}$.

It is readily verified that these open sets in fact form a topology for $X^*$ and that the inclusion $X' \subseteq X^*$ is a topological imbedding.

The family of supports $\mathcal{F}$ is a phi-family [3] provided also:

(IV) each member of $\mathcal{F}$ has a closed neighborhood in $\mathcal{F}$;
(V) each member of $\mathcal{F}$ is paracompact.

Proposition 1. If $\mathcal{F}$ is a phi-family, then $X^*$ is paracompact.

Proof. Let $\mathcal{U}$ be any open cover of $X^*$ and choose $U_\infty \subseteq \mathcal{U}$ with $\in \in U$. Then $X^* - U_\infty \subseteq \mathcal{F}$, so we can find an open set $V$ with $X^* - U_\infty \subseteq V \subseteq X'$, $V \in \mathcal{F}$. Now the sets of the form $U \cap V$, $U \in \mathcal{U}$, cover $V$; since $V$ is paracompact there is a locally finite refinement $\mathcal{U}'$ of this cover of $V$. Now let $\mathcal{W}$ be the family of all sets $U' \cap V$, $U' \in \mathcal{U}'$. Then $\gamma = \mathcal{W} \cup \{U_\infty\}$ is a locally finite open cover of $X^*$, refining $\mathcal{U}$.

To show that $X^*$ is a Hausdorff space, let $x$ and $y$ be distinct points of $X^*$. If $y = \infty$, then $\{x\} \subseteq \mathcal{F}$; we choose a neighborhood $U$ of $x$ with $U \subseteq \mathcal{F}$ and then $U$, $X^* - U$ are disjoint neighborhoods of $x$, $y$. If $x \neq \infty$ and $y \neq \infty$, we choose $U$ as before. If $y \in \mathcal{U}$, then $U$, $X^* - U$ are disjoint neighborhoods of $x, y$. If $y \not\in \mathcal{U}$, then since $\mathcal{U}$ is Hausdorff, we can choose open subsets $V, W$ of $x$ with $x \in V, y \in W$, $V \cap W \cap U = \emptyset$. Then $V \cap U, W \cap (X^* - U)$ are disjoint neighborhoods of $x, y$.

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As examples of families of supports, we cite the following:
(1) The compact subsets of a locally compact space;
(2) The bounded closed subsets of a metric space;
(3) The closed subsets of a space which do not meet a fixed subset $A$. This will be a phi-family if $A$ is closed and the whole space $X$ is paracompact. $X^*$ is the quotient space of $X$ obtained by identifying $A$ to a point. Such a family may be used to define the relative cohomology $H(X, A; \mathcal{F})$ with coefficients in a sheaf $\mathcal{F}$, in a manner described in the next section.

3. Applications to sheaf theory. Let $\mathcal{F}$ be a family of supports for a space $X$ and let $\mathcal{A}$ be a sheaf of rings or modules over $X$. We form the restriction $\mathcal{A}|X'$ of $\mathcal{A}$ to $X'$ and then construct a sheaf $\mathcal{A}^*$ over $X^*$ by defining $\mathcal{A}^*|X' = \mathcal{A}|X'$ and letting the stalk $A^*_x$ of $\mathcal{A}^*$ over $x$ be zero. A neighborhood of $A^*_x$ consists of the zeroes of stalks over a neighborhood of $x$. That the sheaf $\mathcal{A}^*$ is well determined by these data is easily verified. A map $f: \mathcal{A} \to \mathcal{B}$ of sheaves over $X$ determines a map $f^*: \mathcal{A}^* \to \mathcal{B}^*$ of sheaves over $X^*$ in an obvious way, and the following proposition is immediate.

Proposition 2. The correspondence $\mathcal{A} \to \mathcal{A}^*, f \to f^*$ is an exact functor from the category of sheaves over $X$ to that of sheaves over $X^*$.

Proposition 3. If $\mathcal{F}$ is a phi-family and if $\mathcal{A}$ is $\mathcal{F}$-fine, then $\mathcal{A}^*$ is fine.

Proof. Let $\mathcal{U}^*$ be a locally finite open cover of $X^*$. We may assume that $\mathcal{U}^* = \mathcal{U}' \cup \{X^* - F\}$, where $\mathcal{U}'$ is a cover of $X'$ and $F \subseteq \mathcal{F}$, since such covers are clearly cofinal among all covers of $X^*$. Then $\mathcal{U} = \mathcal{U}' \cup \{X - F\}$ is a locally finite $\mathcal{F}$-cover of $X$. If $\{l_i\}$ are endomorphisms of $\mathcal{A}$ making up a partition of unity for $\mathcal{A}$ and $\mathcal{U}$, then the endomorphisms $\{l_i^*\}$ make up a partition of unity for $\mathcal{A}^*$ and $\mathcal{U}^*$. Hence $\mathcal{A}^*$ is fine.

Proposition 4. If $\mathcal{F}$ is any family of supports on a space $X$, and if $\mathcal{A}$ is any sheaf over $X$, then there is a natural isomorphism
$$\Gamma(X^*; \mathcal{A}^*) \approx \Gamma_{\mathcal{F}}(X; \mathcal{A}),$$
where $\Gamma_{\mathcal{F}}(X; \mathcal{A})$ is the module of sections of $\mathcal{A}$ over $X$ with supports in $\mathcal{F}$.

The proof of Proposition 4 is immediate.

Theorem A. Let $\mathcal{F}$ be a phi-family on a space $X$, $\mathcal{A}$ a sheaf over $X$. Then there are natural isomorphisms, for each $p \geq 0$,
$$H^p(X^*; \mathcal{A}^*) \approx H^p_{\mathcal{F}}(X; \mathcal{A}).$$
Proof. The preceding propositions shows that the modules $H^p(X^*; \alpha^*)$ and their associated maps satisfy the Cartan axioms \[3\] for the $\mathcal{F}$-cohomology of $\alpha$.

We do not know whether or not the analog of Theorem A holds for arbitrary families of supports using the Grothendieck cohomology theory of sheaves \[1; 2\]. However, the following result is true.

**Theorem B.** If $\mathcal{F}$ is any family of supports on a space $X$ such that $U\mathcal{F} = X$, and if $\alpha$ is any sheaf on $X$, then there are natural isomorphisms

$$H^p(X^*; \alpha^*) \cong H^p_{\mathcal{F}}(X; \alpha),$$

where the cohomology modules are defined in the sense of Grothendieck.

The proof reduces to the easy verification that, under the assumption on $\mathcal{F}$, if $\alpha$ is injective, then so is $\alpha^*$.

**References**


The University of Chicago