

ON THE ZEROS OF INFRAPOLYNOMIALS FOR PARTLY ARBITRARY POINT SETS¹

MORRIS MARDEN

1. **Introduction.** Let S be a compact point set and let $p(z) = z^n + a_1z^{n-1} + \cdots + a_n$ be an infrapolynomial on S in the sense that it has no underpolynomial on S . (A polynomial $q(z) = z^n + b_1z^{n-1} + \cdots + b_n$ is said to be an underpolynomial of $p(z)$ on S if $q(z) = 0$ on the subset S' of S where $p(z) = 0$ and if $|q(z)| < |p(z)|$ on $S - S'$.) Fejér [1]² has proved that all the zeros of $p(z)$ lie in the convex hull $K(S)$ of S . His theorem is analogous to the theorem of Lucas³ that the zeros of the derivative of a polynomial $f(z)$ lie in the convex hull of the set of the zeros of $f(z)$.

Various other theorems have been proved concerning the location of the zeros of an infrapolynomial. Like Fejér's theorem, many are analogous to known theorems on the location of *all* the zeros of the derivative of a polynomial $f(z)$ when the positions of *all* the zeros of $f(z)$ are prescribed.

However, in the literature there are a number of theorems concerning the location of *some* of the zeros of the derivative of a polynomial $f(z)$ when the positions of only *some* of the zeros of $f(z)$ are prescribed. An example is the theorem of Grace and Heawood⁴ that, if z_1 and z_2 are any two distinct zeros of an n th degree polynomial $f(z)$, at least one zero of the derivative of $f(z)$ lies in the circle with center at $(1/2)(z_1 + z_2)$ and radius $(1/2)|z_1 - z_2| \cot(\pi/n)$. Another example is the theorem due to Marden⁵ that, if K zeros ($2 \leq K \leq n$) of an n th degree polynomial $f(z)$ lie in a circle of radius R , at least $K - 1$ zeros of $f'(z)$ lie in the concentric circle of radius $R \csc[\pi/2(n - K + 1)]$.

It is the object of the present note to develop for infrapolynomials results analogous to these theorems of Grace-Heawood and Marden. The results involve the location of *some* of the zeros of an infrapoly-

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² A number enclosed in a square bracket refers to the bibliography at the close of this note.

³ See [4, pp. 14-16].

⁴ See [4, pp. 84-85].

⁵ See [3, p. 364] or [4, p. 90].

nomial for a set S comprised of points, only *some* of which have prescribed positions.

2. THEOREM 1. Let $p(z) = z^n + a_1z^{n-1} + \dots + a_n$ be an *infrapolynomial* on the set $S = S_0 + S_1$ where S_0 is a compact point set (finite or infinite) and S_1 is a set of k points, $0 \leq k \leq n$. Let T_0 be the set comprised of all points from which S_0 subtends an angle of at least $\pi/(k+1)$. If $p(z) \neq 0$ on S , then $p(z)$ has at most k zeros outside T_0 irrespective of the location of S_1 .

PROOF. We shall make use of the following result due to Fekete [2].

FEKETE'S THEOREM. Let $p(z) = z^n + a_1z^{n-1} + \dots + a_n$ be an *infrapolynomial* on S . If $p(z) \neq 0$ on S , then there exist $(m+1)$ positive constants λ_j with $\lambda_0 + \lambda_1 + \dots + \lambda_m = 1$ and $(m+1)$ points z_j in S such that $p(z)$ is a factor of the polynomial

$$\Omega(z) = \omega(z) \sum_{j=0}^m [\lambda_j/(z - z_j)], \quad n \leq m \leq 2n,$$

where

$$\omega(z) = (z - z_0)(z - z_1) \dots (z - z_m).$$

It follows that the zeros of $p(z)$ satisfy an equation of the form

$$(2.1) \quad \sum_{j=0}^m \lambda_j/(z - z_j) = 0.$$

Let us suppose that $p(z)$ has $k+1$ zeros, Z_0, Z_1, \dots, Z_k outside T_0 . Let $(2h)$ be the shortest distance of any of the points Z_j to the boundary of T_0 . The zeros of (2.1) are continuous functions of the λ_j . If we choose ϵ as a sufficiently small number, $0 < \epsilon < h$, we can find a $\delta > 0$ such that for rational numbers ρ_j with $|\rho_j - \lambda_j| < \delta, j = 0, 1, \dots, m$, the equation

$$(2.2) \quad \sum_{j=0}^m [\rho_j/(z - z_j)] = 0$$

has roots $\zeta_0, \zeta_1, \dots, \zeta_k$ with $|\zeta_j - Z_j| < \epsilon$ for $j = 0, 1, \dots, m$. Thus the points $\zeta_0, \zeta_1, \dots, \zeta_k$ also lie outside T_0 .

Let us next choose a positive, sufficiently large integer N so that each of the quantities $\nu_j = \rho_j N$ is an integer. The equation

$$(2.3) \quad \sum_{j=0}^m [\nu_j/(z - z_j)] = 0$$

which has also the roots $\zeta_0, \zeta_1, \dots, \zeta_k$, is satisfied by those zeros of

the derivative of the polynomial $f(z) = \prod_{j=0}^m (z - z_j)^{n_j}$ which are distinct from the z_j .

Since all that is known about the z_j is that each lies in S , some of them may be points of S_1 . But the set S_1 contains only k points so that at least $m+1-k$ of the z_j do not lie in S_1 . Let us, if necessary, relabel the z_j so that the points z_0, z_1, \dots, z_{m-k} all lie in S_0 . Now any set of $m+2$ distinct points comprised of $m-k+1$ zeros z_0, z_1, \dots, z_{m-k} of $f(z)$ and $k+1$ zeros $\zeta_0, \zeta_1, \dots, \zeta_k$ of $f'(z)$, are known⁶ to satisfy an identity of the form

$$(2.4) \quad \sum M_{j_0 j_1 \dots j_k} [(\zeta_0 - z_{j_0})(\zeta_1 - z_{j_1}) \dots (\zeta_k - z_{j_k})]^{-1} = 0$$

where the j_0, j_1, \dots, j_k run independently through the values $0, 1, \dots, m-k$ and the numbers $M_{j_0 j_1 \dots j_k}$ are all positive.

Since $\zeta_0, \zeta_1, \zeta_k$ all lie outside T_0 , points $\tau_0, \tau_1, \dots, \tau_k$ may be found in S_0 so that for each i

$$(2.5) \quad 0 < \arg [(\zeta_i - \tau_i)/(\zeta_i - z_{j_i})] < \pi/(k+1).$$

If the left-side of (2.4) is multiplied by $(\zeta_0 - \tau_0)(\zeta_1 - \tau_1) \dots (\zeta_k - \tau_k)$, each term in the resulting sum has, due to (2.5), the property

$$0 < \arg \prod_{i=0}^k [(\zeta_i - \tau_i)/(\zeta_i - z_{j_i})] < \pi.$$

Hence the sum in (2.4) cannot vanish and so at most k zeros of any infrapolynomial for S can lie outside of T_0 .

3. Corollaries. Fejér's Theorem, stated in §1, is the special case of Theorem 1 corresponding to $k=0$. Other immediate corollaries are the following:

COROLLARY 1. *Let S_0 be a compact point set all of the points of which lie in a circle C of radius R . Let S be the point set comprised of S_0 and k additional points, the latter having arbitrary positions in the complex plane. Then any infrapolynomial $p(z) = z^n + a_1 z^{n-1} + \dots + a_n \neq 0$ on S has at least $n-k$ zeros within the concentric circle of radius $R \csc [\pi/2(k+1)]$.*

COROLLARY 2. *Let S_0 be a compact set situated on the line segment joining the points $z=a$ and $z=b$ of the real axis. Let S be comprised of S_0 and k additional points situated at random in the plane. If $p(z) = z^n + \dots$ is any nonvanishing infrapolynomial on S , then at least $n-k$ zeros of $p(z)$ lie in region bounded by the two circles with radius $L \csc \pi/(k+1)$ and centers $z = c \pm iL \cot [\pi/(k+1)]$ where $c = (1/2)(a+b)$ and $L = (1/2)|a-b|$.*

⁶ See [3, pp. 356-364].

4. Generalization. Theorem 1 may be extended to the case that $p(z)$ has some zeros on S , as follows.

THEOREM 2. *In the notation of Theorem 1, if $p(z)$ has ν zeros on S_1 where $0 \leq \nu \leq \min(k, n-k)$, then $p(z)$ has at most $\nu+k$ zeros (counted with their multiplicities) outside T_0 irrespective of the location of S_1 .*

PROOF. Let us assume that $p(z)$ has zeros both on S_0 and S_1 and denote these by $\xi_1, \xi_2, \dots, \xi_\mu$ and $\zeta_1, \zeta_2, \dots, \zeta_\nu$ respectively where $0 \leq \mu \leq n, 0 \leq \nu \leq n, \mu + \nu \leq \min(k, n-k)$. Let us write

$$p(z) = (z - \xi_1) \cdots (z - \xi_\mu)(z - \zeta_1) \cdots (z - \zeta_\nu)p_1(z)$$

where $p_1(z) \neq 0$ on S . Then, relative to polynomials of degree $n - \mu - \nu$, $p_1(z)$ is an infrapolynomial on S , since otherwise $p_1(z)$ would have an underpolynomial $q_1(z)$ and thus $p(z)$ would have the underpolynomial $q(z) = [p(z)/p_1(z)]q_1(z)$.

On applying Theorem 1, we learn that $p_1(z)$ has at least $n - \mu - \nu - k$ zeros on T_0 . But, as $S_0 \subset T_0$, $p(z)$ has as zeros in T_0 the points ξ_j as well as the zeros of $p_1(z)$. Thus $p(z)$ has at least $n - \nu - k$ zeros in T_0 , as was to be proved.

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UNIVERSITY OF WISCONSIN-MILWAUKEE