A NOTE ON PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITHOUT DAMPING

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We consider the differential equation

\begin{equation}
\dot{x} + g(x) = p(t),
\end{equation}

where, in addition to satisfying conditions which insure existence and uniqueness of solutions, \(g(x)\) and \(p(t)\) satisfy the following conditions:

(i) there exist positive constants \(\tau\) and \(k\) such that for all \(t\), \(p(t) = p(t+\tau)\) and \(|p(t)| < k\);

(ii) \(g(x)\) is continuous, increasing, \(g(0) = 0\), and for \(|x|\) sufficiently large, \(|g(x)| > k\).

We show that for \(\tau\) sufficiently small, Equation (1) has a solution \(x(t)\) of period \(\tau\). Our method of proof consists of showing that a certain mapping of the phase plane of Equation (1) has a fixed point, not by use of the Brouwer theorem, but by a somewhat more general result concerning the index of the bounding curve of a simply-connected region relative to a vector field induced by the mapping; cf. [1, p. 337].

We consider Equation (1) in terms of the system:

\begin{equation}
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= p(t) - g(x),
\end{aligned}
\end{equation}

the solutions \((x(t), y(t))\) of which define the so-called phase trajectories of the system. Let \(x = a\) be the root of \(g(x) = -k\), and \(x = b\) be the root of \(g(x) = k\). That these roots exist and are unique follows immediately from (ii). We define

\[ G(x) = \int_0^x g(v)dv \]

and using (ii), we observe that the graph of \(y = G(x)\) has the following properties:

(iii) it is tangent to the \(x\)-axis at the origin and is everywhere concave up;

(iv) its slope \(G'(x) = g(x)\) is such that \(-k < G'(x) < k\) for \(a < x < b\), \(G'(x) > k\) for \(x > b\), and \(G'(x) < -k\) for \(x < a\).

Hence, there clearly exists a positive constant \(C_1\) such that the line \(y = kx + C_1\) intersects the graph of \(y = G(x)\) at exactly two points whose abscissas \(r_2\) and \(r_3\) are such that \(r_2 < a < b < r_3\). Similarly there exists a positive constant \(C_2\) such that the line \(y = -kx + C_2\) intersects the graph of \(y = G(x)\) at exactly two points whose abscissas \(r_1\) and \(r_4\) are such that \(r_1 < r_2 < r_3 < r_4\).

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We denote by $\Gamma_1$ the open arc defined by $y^2/2 + G(x) = kx + C_1$, $y > 0$, by $\Gamma_2$ the open arc defined by $y^2/2 + G(x) = -kx + C_2$, $y < 0$, and observe that the end points of these arcs are at $(r_2, 0)$, $(r_3, 0)$ and $(r_1, 0)$, $(r_4, 0)$ respectively.

**Lemma 1.** The slope of any phase trajectory of (2) at a point of either $\Gamma_1$ or $\Gamma_2$ is less than the slope of the curve $\Gamma_1$ or $\Gamma_2$ at that point.

**Proof.** We prove the lemma only for a point $(x_0, y_0)$ on $\Gamma_1$; the proof for a point on $\Gamma_2$ is entirely similar, and is omitted. Clearly the slope of $\Gamma_1$ at $(x_0, y_0)$ is given by $(k - g(x_0))/y_0$, while the slope of the trajectory of (2) at $(x_0, y_0)$ for $t = t_0$ is given by $(p(t_0) - g(x_0))/y_0$, which is clearly less than the slope of $\Gamma_1$ there. This proves the lemma.

We now denote by $P_1$, $P_2$, and $P_3$ the points $(r_2, 0)$ and $(r_3, 0)$ respectively; by $Q$ the intersection of the line $x = r_3$ with $\Gamma_2$, and by $R$ the closed region bounded by the curves $\Gamma_1$, $\Gamma_2$ from $P_1$ to $Q$, and the line segments $P_1P_2$, $P_3Q$. We now prove another simple

**Lemma 2.** Let $(x_0, y_0)$ be a point on the boundary of $R$, and let $(x(t), y(t))$ be the solution of (2) for which $x(0) = x_0$, $y(0) = y_0$. Then the trajectory of this solution can only leave $R$ for some $t_0 > 0$, if $r_1 < x(t_0) \leq r_2$, and $y(t_0) = 0$.

**Proof.** Due to Lemma 1, we need only prove that if $(x_0, y_0)$ is on the segment $P_3Q$, then the trajectory at $(x_0, y_0)$ for some arbitrary $t = t_0 > 0$ passes first into $R$ as $t$ increases from $t_0$. This is obvious if $(x_0, y_0)$ is $Q$, since $Q$ is on $\Gamma_2$; it is also obvious for $(x_0, y_0)$ between $P_3$ and $Q$ since then $y_0 < 0$, and hence $x(t_0) = y(t_0) < 0$. Finally if $(x_0, y_0)$ is $P_3$, then since $y(t_0) = -g(x(t_0)) + p(t_0) < 0$, $y(t) < 0$ for $t > t_0$, $t - t_0$ sufficiently small; hence $x(t) < x(t_0)$ also for $t > t_0$, $t - t_0$ sufficiently small.

This essentially proves the lemma.

We now define $M = \left| 2(G(a) + ka - C_2) \right|^{1/2}$, $m = \left| 2(G(a) - ka - C_1) \right|^{1/2}$, $d = a - r_2$, and $k = \max_{r_1 \leq x \leq r_2} (-g(x) + k)$ and prove the following:

**Theorem.** In terms of the definitions above, if $\tau < \min (d/M, m/K)$, then system (2), hence Equation (1), has a solution of period $\tau$.

**Proof.** We define the mapping $T$ of the $(x, y)$ plane into itself as follows: $T(x_0, y_0) = (x_1, y_1)$ where $x_1 = x(\tau)$, $y_1 = y(\tau)$, and $(x(t), y(t))$ is the solution of (2) such that $x(0) = 0$, $y(0) = 0$. We show that this mapping has a fixed point; this will prove the theorem; cf. [1, p. 270].

We consider the vector field defined by this mapping, and will show that the index of the boundary of $R$ relative of this field (cf. [1, p. 337]) is $+1$; this will prove that $R$ contains a critical point of this field; i.e., that the mapping $T$ has a fixed point in $R$.
Let \((x_0, y_0)\) be a point of \(\Gamma_1\); we show first that \(T(x_0, y_0)\) is in \(R\). For if not, there exists by Lemma 2 a smallest \(t_1, 0 < t_1 < \tau\), such that \(y(t_1) = 0, r_1 \leq x(t_1) \leq r_2\); here as throughout this proof \((x(t), y(t))\) is the solution of (2) used in the definition of \(T\). Suppose first \(x_0 = a\); then clearly for \(0 < t < t_1, \left| y(t) \right| < M\). Since \(x(t) = y(t)\),

\[
\left| x(t_1) - x_0 \right| = \left| \int_0^{t_1} y(t) \, dt \right| < M \tau.
\]

However, \(\left| x(t_1) - x_0 \right| \geq a - r_2 = d\); i.e., \(d < \tau M\), a contradiction. If \(r_2 < x_0 < a\), then since \(\dot{y} = -g(x) + p(t) > 0\) for \(x = x(t) < a\), there exists a \(t_0 > 0\) such that \((x(t), y(t))\) is in \(R\) for \(0 \leq t \leq t_0\), and \(x(t_0) = a\). From here, the argument toward a contradiction proceeds as above, and we omit it. In fact, if \((x_0, y_0)\) is on either the segment \(P_3Q\) or the curve \(\Gamma_2\) from \((a, -M)\) to \(Q\), the same argument, in essence, applies to show that \(T(x_0, y_0)\) is in \(R\).

If \((x_0, y_0)\) is on \(\Gamma_2\) while \(x < a\), or on the segment \(P_1P_2\), then if \(T(x_0, y_0) = (x_1, y_1)\) is not in \(R\), we show finally that \(r_1 \leq x_1 \leq a, 0 \leq y_1 \leq m\). In any case, there exists \(t_0, 0 \leq t_0 < \tau\), such that \(r_1 \leq x(t_0) \leq r_2\), \(y(t_0) = 0\). If \(x(t) < a\) for \(t_0 < t < \tau\), then clearly \(x(t) > r_1\) for these values of \(t\). Assume that for \(t_1, t_0 < t_1 < \tau\), we have \(y(t_1) > m\). Then since \(\dot{y} = -g(x) + p(t)\), we have

\[
y(t_1) - y(t_0) = \int_{t_0}^{t_1} (-g(x(t)) + p(t)) \, dt < K \tau.
\]

But \(y(t_1) - y(t_0) = y(t_1) > m\); hence \(m < K \tau\), a contradiction, and we conclude that \(x(t_2) = a\) for some \(t_2, t_0 < t_2 < \tau\), while \(0 < y(t_2) < m\). Since \(M > m\), and \(x(t_2) - x(t_0) > d\), we easily arrive at a contradiction as before by integrating \(\dot{x}(t) = y(t)\) from \(t_0\) to \(t_2\); we omit the details.

If the vector field \((x, y) \rightarrow T(x, y)\) has a critical point on the boundary of \(R\), then by definition, the mapping has a fixed point there and there is nothing more to prove. We therefore assume that as \((x, y)\) moves in a counter-clockwise circuit around the boundary of \(R\), the vector at each point of the boundary is nonzero; since it is also a continuous function of \((x, y)\) and in view of the location of the terminal point \(T(x, y)\) for \((x, y)\) on the boundary of \(R\) as established in the proof of this theorem, it follows that the index of this circuit relative to this field is +1. This completes the proof of the theorem.

Reference