SOME SEMIGROUPS ON THE TWO-CELL

ANNE LESTER

A topological semigroup is a nonvoid, Hausdorff space, $S$, together with a continuous, associative multiplication defined on $S$. Here the term semigroup will always denote a topological semigroup. In [4], Wallace and Koch have shown that if the circle, $B$, is a semigroup such that $B^2 = B$, then either $B$ is a group, or the multiplication on $B$ is of the trivial kind $xy = x$, or $xy = y$. In [6], Mostert and Shields give a description of a semigroup on the two-cell where the boundary of the two-cell relative to the plane is a group. The main results of this paper will be a description of a semigroup on the two-cell where the multiplication satisfies the following two conditions: one; for $x$ and $y$ in the boundary of the two-cell relative to the plane $xy = x$, and two; there is a zero but no other idempotent in the interior of the two-cell. The following theorem will be proved:

**Theorem A.** Let $S$, topologically the two-cell, be a semigroup with zero in the interior. Assume for $x$ and $y$ in $B$, the boundary of $S$ relative to the plane, $xy = x$. Suppose, moreover, that $S$ has no idempotents, except zero, in the interior. Then there exists an I-semigroup $T$ in $S$ such that $S = BT$. Also for $e$ and $f$ in $B$, $t$ and $s$ in $T$ $(es)(ft) = e(st)$; and if $es = ft$ then $s = t$.

There is, as will be obvious from the proof, a left-right dual of this theorem. Due to the length of the proof of this theorem, which will be given in a sequence of lemmas and theorems, some definitions and background material have been omitted. The reader is referred to [6] for any definitions and results not given here.

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A semigroup $P$ will be said to satisfy condition $(\alpha)$ if (i) there exists a set $A$ in $P$ such that $A \subseteq f(A)$, where $f$ is the function defined by $f(x) = x^2$ for all $x$ in $P$; and (ii) there exists $y$ in $A$ such that $y \neq y^2$. In the following discussion $P$ will be assumed to be a compact semigroup which satisfies condition $(\alpha)$. For an element $p$ in $A$ such that

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$p \neq p^2$, $s(p)$ will denote a fixed sequence of elements $p_n$, $n = 0, 1, 2, \ldots$ with the properties that (i) $p_0 = p$, and (ii) $p_n^2 = p_{n-1}$. For positive integers $n$ and $j$, let $Z_{n,j}(s(p))$ be defined as follows:

$$Z_{n,j}(s(p)) = \left\{ p_m : \frac{i}{2m} \leq \frac{j}{2n} \text{ and } p_m \in s(p) \right\}^*,$$

where $A^*$ denotes the topological closure of a set $A$. For real, non-negative $t$, let $Z_t(s(p)) = \bigcap \{Z_{n,j}(s(p)) : j/2^n > t \}$. Also let $\delta(Z_t(s(p)) = Z_t(s(p)) \setminus \bigcup_{r < t} Z_r(s(p))$, where $A \setminus B$ denotes the set theoretic difference of $A$ by $B$. In the following discussion consider the element $p$ in $A$ and the sequence $s(p)$ fixed and let $Z_0(s(p))$ be denoted by $Z$.

**Lemma 1.** $Z = \bigcap_{t > 0} \{Z_t(s(p))\}$ and $Z$ is a compact, commutative group.

**Proof.** Clearly $Z = \bigcap_{t > 0} \{Z_t(s(p))\}$ and $Z$ is nonvoid since $P$ is compact and $Z$ is the intersection of a tower of closed subsets of $P$. Since $Z$ is closed it is also compact. Using the fact that multiplication is continuous in $P$ and that the sequence $s(p)$ was chosen so that $p_n^2 = p_{n-1}$, it can be easily shown that $Z$ is a commutative subsemigroup of $P$ and for $a$ in $Z$, $p_n$ in $s(p)$ that $p_n \in a P \cap Pa$. From this last fact it follows that any idempotent in $Z$ acts as an identity for the sequence $s(p)$, and therefore for all of $Z$. Since $Z$ is a compact subsemigroup of $P$ it contains at least one idempotent. But any idempotent in $Z$ acts as an identity, hence $Z$ can contain at most one idempotent. Thus $Z$ is a group, [9].

**Lemma 2.** If $y$ and $z$ belong to $\delta(Z_t(s(p)))$, there exists $g$ in $Z$ such that $y = gz$.

**Proof.** Since $y$ and $z$ are in $\delta(Z_t(s(p)))$, $y = \lim p_{m_n}^i$ and $z = \lim p_{k_n}^j$ where $p_{m_n}$ and $p_{k_n}$ are subsequences of $s(p)$ and $i_n/2^{m_n} \rightarrow t = j_n/2^{k_n}$. We may assume that $i_n/2^{m_n} \geq j_n/2^{k_n}$ for all $n$. Let $r_n/2^q_n = i_n/2^{m_n} - j_n/2^{k_n}$. Now $r_n/2^q_n$ converges to zero, therefore $\{p_{q_n}^n\}$ clusters in $Z$. Since a subsequence of $\{p_{q_n}^n\}$ converges to some $g$ in $Z$, let us assume that the sequence converges to $g$. Hence we have, $y = \lim p_{m_n}^i = (\lim p_{q_n}^n)(\lim p_{k_n}^j) = gz$.

**Theorem 1.** Let $P$ be a compact semigroup satisfying condition (a). For $p$ in $A$ such that $p \neq p^2$ let $Z(s(p))$ be defined as above. Then there exists a one-parameter semigroup $\sigma$ from $[0, 1]$ to $Z_1(s(p))$, such that $\sigma(0) = e$, the identity of $Z(s(p))$, and $\sigma(1)g = p$ for some $g$ in $Z(s(p))$.

**Proof.** The proof of this theorem is long and is divided into a sequence of short steps.
(i) Since \( e \in Z(s(p)) \), there is a sequence \( \{ p_{mn}^f \} \) such that \( e = \lim p_{mn}^f \) and \( i_n/2^{m_n} \) converges to zero. For notational reasons, let \( r_n = i_n/2^{m_n} \) and \( p_{mn}^f = p^f_n \). For each \( n \), define \( \theta(n) = \left\lfloor \frac{1}{r_n} \right\rfloor \), where \( \lfloor a \rfloor \) is the least integer greater than or equal to \( a \). From the definition of \( \theta(n) \) and the fact that \( r_n \to 0 \), it follows that \( \theta(n) r_n \to 1^+ \).

For each real number \( s \), such that \( 0 < s \leq 1 \), define a function \( f_s \) from \( J \), the positive integers, to \( Z_1(s(p)) \) by \( f_s(n) = \rho^{(\theta(n))} r_n \). For each \( s \), \( f_s \) is a continuous function and therefore has a unique extension, \( f_s^* \), to \( \beta(J) \), the Stone-Čech compactification of \( J \). Let \( \alpha \in \beta(J) \setminus J \) be chosen and considered fixed in the following discussion.

For each real \( s \), \( 0 < s \leq 1 \), define \( \tau(s) = f_s^*(\alpha) \) and \( \tau(0) = e \). From the definition of \( \tau \) it can be shown that for \( s, t \) and \( s + t \) in \( [0, 1] \), \( \tau(s + t) = \tau(s) \tau(t) \). Also from the definition of \( \tau \) and the fact that \( [\theta(n) t] r_n \to t^+ \), it follows that \( \tau(t) \in Z_1(s(p)) \) and \( \tau(t) \) does not belong to \( Z_1(s(p)) \) for any \( t < \tau(t) \), that is, \( \tau(t) \in Z_1(t(s(p))) \) for \( 0 < t \leq 1 \).

(ii) \( G \), defined by \( G = \bigcap_{n>0} \{ \tau([0, n]) \} \), is a group with identity \( e = \tau(0) \). Clearly \( e \in G \) and \( G \subseteq Z(s(p)) \). Thus it remains to show that \( G^2 \subseteq G \), since \( G \) is compact and contains only one idempotent which acts as an identity. By a straightforward argument using the facts that multiplication is continuous in \( P \) and that for \( s, t \) and \( s + t \) in \( [0, 1] \), \( \tau(s + t) = \tau(s) \tau(t) \), it follows immediately that \( G^2 \subseteq G \).

(iii) Let \( Q \) be defined by \( Q = (\tau([0, 1/2]))^* \). \( Q \) is a compact, adequate local semigroup with maximal subgroup \( G \). This statement follows almost immediately from the facts that if \( x \) is in \( Q \) then \( x = \tau(a) g \) for some \( a \) such that \( 0 \leq a \leq 1/2 \), and some \( g \in G \). To show the latter let us assume that \( x \) belongs to \( Q \). Then there is a sequence \( a_n \) in \( [0, 1/2] \) such that \( x = \lim \tau(a_n) \). Since \( a_n \in [0, 1/2] \) for each \( n \), there is a real number \( t \) in \( [0, 1/2] \) such that \( t = \lim a_m \), where \( a_m = a_{n_j} \) is some subsequence of \( a_n \). We may assume that \( a_m \leq t \) for each \( m \), or \( a_m \geq t \) for each \( m \). If \( a_m \leq t \), let \( b_m = t - a_m \). Then \( \tau(t) = \tau(a_m) \tau(b_m) \) and \( b_m \to 0 \). Hence \( \tau(t) = \lim \tau(a_m) \lim \tau(b_m) \). That is \( \tau(t) = x g \) for some \( g \in G \). Thus \( x = \tau(t) g^{-1} \). The case for \( a_m \geq t \), for each \( m \), is similar.

Now suppose \( x \) and \( y \) are in \( Q \) and \( xy = e \). By the preceding argument \( x = \tau(a) g \) and \( y = \tau(b) h \) for some \( a \) and \( b \) in \( [0, 1/2] \), \( g \) and \( h \) in \( G \). Since \( xy = e \), by (ii) we may conclude that \( a + b = 0 \), hence \( a = b = 0 \). Thus \( x = g \) and \( y = h \) and both belong to \( G \). This concludes the proof of the above statement.

It is not difficult to show that \( Q \) contains no idempotents except \( \tau(0) \), hence by [5], there is a one-parameter semigroup \( \gamma \) from \( [0, 1] \) to \( Q \) such that \( \gamma(t) \in G \) if and only if \( t = 0 \). Reparametrize \( \gamma \) so that \( \gamma(t)/G = \tau(t)/G \) and denote the new one-parameter semigroup by \( \sigma \). This gives the first part of Theorem 1. To obtain \( p = \sigma(1) g \) for some
g in \( Z(s(p)) \), we need only to note that \( p \) and \( \sigma(1) \) both belong to \( \delta(Z(s(p))) \) and apply Lemma 2.

**Proof of Theorem A.** Throughout this proof \( S \) will denote a semigroup satisfying the hypothesis of Theorem A.

**Lemma 3.** For each \( a \) in \( S \), there exists an \( x \) in \( S \) such that \( x^2 = a \).

**Proof.** Define \( f: S \to S \) by \( f(x) = x^2 \). \( f \) is a continuous function and for each \( e \) in \( B \), the boundary of \( S \), \( f(e) = e^2 = e \). If \( f \) does not map \( S \) onto \( S \), then it would be possible to retract \( S \) onto \( B \) which presents a contradiction. Thus \( f \) maps \( S \) onto \( S \) and the lemma is proved.

Let \( p \neq 0 \) be in \( S^0 \), the interior of \( S \), and let \( s(p) \) be a fixed sequence of \( 2^n \)th roots of \( p \) as defined above. Let \( e \) denote the identity of \( Z(s(p)) \), which, by the hypothesis on \( S \), belongs to \( B \). Denote by \( \sigma \) the one-parameter semigroup as constructed in Theorem 1 such that \( \sigma(0) = e \).

**Lemma 4.** For \( t \) in \( (1, \infty) \), define \( \tau(t) = \sigma(1)^n \sigma(a) \), where \( t = n + a \), \( n \) an integer and \( 0 \leq a < 1 \). For \( t \) in \([0, 1]\) define \( \tau(t) = \sigma(t) \). \( \tau \) so defined is a continuous homomorphism from \( R \), the additive semigroup of reals \( \geq 0 \), into \( S \) and \( (\tau[0, \infty))^* \) is a compact, connected, commutative sub-semigroup with identity \( e \).

**Proof.** The proof of this lemma makes frequent use of the fact that \( \sigma(1) = \sigma(t) \sigma(1 - t) \) for any \( t \) in \([0, 1]\). Using this it is easy to see that \( \sigma(1)^n \sigma(t) = \sigma(t) \sigma(1)^n \) and hence from the definition of \( \tau \), that \( \tau(a + b) = \tau(a) \tau(b) \) for all real \( a \) and \( b \). This proves that \( \tau \) is a homomorphism from \( R \) into \( S \). Since \( \tau[0, \infty) \) is a commutative, connected subsemigroup of \( S \), it follows that \( (\tau[0, \infty))^* \) is again one such.

In the following lemma, \( J(x) = x \cup xS \cup Sx \cup SxS \), \( J_x = \{ y: J(y) = J(x) \} \) and \( I(x) = J(x) \setminus J_x \).

**Lemma 5.** Let \( C \) be a continuum that is a semigroup with zero. Let \( A = A^* \) be a subset of \( C \) such that for some \( x_0 \) in \( C \) the following holds:

1. \( A^* \setminus A^0 \subseteq J_{x_0} \);
2. \( J_{x_0} \neq \{ x_0 \} \); and
3. \( A^0 \) is nonvoid and is contained in \( I(x_0) \). Then the zero for \( C \) belongs to \( A^0 \).

**Proof.** In [9], Wallace proves that if \( C \) is a continuum and \( J_{x_0} \neq \{ x_0 \} \) then \( I(x_0)^* = J(x_0) \) and \( I(x_0) \) is connected. By the above hypothesis on \( A \) we have \( I(x_0) \cap A = (I(x_0) \cap A^0) \cup (I(x_0) \cap A^* \setminus A^0) = I(x_0) \cap A^0 \). This implies that \( I(x_0) \cap A^0 \) is both open and closed in \( I(x_0) \), a connected set. Thus either \( I(x_0) \cap A^0 \) is void or \( I(x_0) \cap A^0 = I(x_0) \). If \( I(x_0) \cap A^0 \) is void, since \( J(x_0) = I(x_0)^* \), we have \( J(x_0) \subseteq C \setminus A^0 \), a contradiction by condition 3. Hence the zero for \( C \) belongs to \( I(x_0) \) which is contained in \( A^0 \) and the lemma is proved.
COROLLARY. Let C be a continuum that is a semigroup with zero. Let $A = A^*$ be a subset of C such that: (1) $A^* \setminus A^0$ is a nontrivial group with identity $u$; and (2) $A^0$ is nonvoid and contained in $J(u)$. Then the zero for $C$ belongs to $A^0$.

PROOF. This follows immediately from the above lemma letting $x_0 = u$.

**Lemma 6.** Let $S$ be as defined above. The zero of $S$ belongs to $(\tau [0, \infty))^*$. 

**Proof.** Let $(\tau [0, \infty))^*$ be denoted by $T$ and assume that $0 \in T$. First let us note that if $f$ is any other idempotent in $T$, $f$ belongs to $B$ and therefore $e = ef = fe = f$ since $T$ is commutative. It follows from this fact that $T$ is a group since it contains at most one idempotent which acts as an identity.

(i) *To show $T$ is one-dimensional.* If $T$ were two-dimensional, then $T$ would contain a two-cell, [1, p. 239], and would, therefore, be both opened and closed in $S$ which is a contradiction. Hence $T$ is a one-dimensional group.

(ii) *To show $T$ is a circle.* First let us note that for any $x \neq 0$ in $S^0$ there exists $f \in B$ such that $x = xf = fx$. This follows from the fact that for any sequence $s(x)$ of $2^n$th roots of $x$ there is an $f$ in $Z(s(x))$ which acts as an identity for $s(x)$. From this fact it follows that $e$ is a right unit for $S$, since for any $x$ in $S$, $xe = (xf)e = x(he) = xf = x$, where $f$ is as above. Thus $T$ can be considered as a compact group acting on $S$. Hence $T$ is locally connected, since $T$ is a one-dimensional orbit [1, p. 248]. Thus $T$ is a circle group, since the only connected, one-dimensional groups are the circle group and the solenoid, and the latter is not locally connected.

(iii) Since $T$ is a circle, $S \setminus T = P \cup Q$, where $P$ and $Q$ are open, disjoint, nonempty sets. $P^* \setminus P^0 = T = Q^* \setminus Q^0$, hence $P^*$ and $Q^*$ satisfy the hypothesis for the corollary of Lemma 5, since $P \subseteq J(e)$ and $Q \subseteq J(e)$. Thus $0 \in P \cap Q$, which is a contradiction. From this last contradiction it follows that 0 belongs to $T$.

**Lemma 7.** For all real $t$, $0 < t < \infty$, \( \lim_{n \to \infty} \tau(t)^n = 0 \).

**Proof.** In the following $\Gamma(a) = \{ a, a^2, a^3, \cdots \}$*. Since the only idempotents in $T$ are 0 and e, there exists at least one $t_0$ such that $0 \in \Gamma(\tau(t_0))$ for otherwise $T$ would be a group which is a contradiction. Now for $a > t_0$, $a = t_0 + b$ for some real $b$, hence $\tau(a) = \tau(t_0)\tau(b)$. Thus $\tau(a)^n = \tau(t_0)^n\tau(b)^n$ for all positive integers $n$. Since $0 \in \Gamma(\tau(t_0))$, \( \lim_{n \to \infty} \tau(t_0)^n = 0 \). Now either $e \in \Gamma(\tau(b))$ or $0 \in \Gamma(\tau(b))$, but in either case, there is a subsequence $\tau(b))^n$ converging to an idempotent. From this it follows that $\lim_{n \to \infty} \tau(a)^n = \lim_{n \to \infty} \tau(t_0)^n \lim_{n \to \infty} \tau(b)^n = 0 \lim_{n \to \infty} \tau(b)^n = 0$. 

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Therefore \( \lim r(a)^n = 0 \) for all real \( a > t_0 \). For \( 0 < b < t_0 \), there exists an integer \( n \) such that \( t_0 \leq nb \), or \( t_0/n \leq b \). Since \( \tau(t_0/n)^n = \tau(nt_0/n) = \tau(t_0) \), we have \( \Gamma(\tau(t_0)) \subseteq \Gamma(\tau(t_0/n)) \) and therefore \( 0 \in \Gamma(\tau(t_0/n)) \). Since \( t_0/n \leq b \), the preceding argument applies, therefore \( \lim \tau(b)^n = 0 \). Hence the lemma is true.

**Lemma 8.** \( T = \tau[0, \infty) \cup 0 \).

**Proof.** The proof of this lemma follows immediately from the proof in [3, p. 18] for a more general situation.

**Lemma 9.** \( T \) is an \( I \)-semigroup and there exists \( t_0 \), either finite or equal to \( \infty \) such that \( T = \tau[0, t_0] \) and \( \tau \) is one-one on \( [0, t_0] \).

**Proof.** Clearly \( T \) is an arc with endpoints \( e \) and \( 0 \). Since \( e \) acts as an identity for \( T \) and \( 0 \) functions as a zero, \( T \) is an \( I \)-semigroup. If there exists a finite \( t_0 > 0 \) such that \( \tau(t) = 0 \), let \( t_0 = \inf \{ t: \tau(t) = 0 \} \). If there exists no \( t \) such that \( \tau(t) = 0 \), let \( t_0 = \infty \).

If \( \tau \) is not one-one on \([0, t_0]\), there must exist real numbers \( r \) and \( s \) in \([0, t_0]\) such that \( r < s \) and \( \tau(r) = \tau(s) \). Under this assumption, by a straightforward argument, making use of the fact that for any positive integer \( n \), \( \tau(ns - (n - 1)r) = \tau(r) \), it can be shown that \( T \subseteq \tau[0, s] \), a contradiction since \( s < t_0 \).

**Lemma 10.** For \( s \) in \( S \), \( s = fj \) where \( f \in B \) and \( j \in T \). Moreover, if \( fj = hk \) for some \( f, h \in B \) and \( j, k \in T \), then \( j = k \).

**Proof.** Let \( \theta: B \times T \to S \) be defined by \( \theta(g, t) = gt \). If \( \theta \) is not onto there exists \( p \) in \( S \) such that \( p \notin \theta(B \times T) \). Let \( \delta: S \setminus p \to B \) be such that \( \delta \) is continuous and for \( x \) in \( B \), \( \delta(x) = x \). Now define \( \alpha: B \times T \to B \) by \( \alpha(f, t) = \delta(\theta(f, t)) \). Since \( T \) is compact and connected it follows that the identity function on \( B \) is homotopic to a constant function. For \( \alpha(x, e) = \delta(\theta(x, e)) = \delta(xe) = \delta(x) = x \) for any \( x \) in \( B \), and \( \alpha(x, 0) = \delta(0) = h_0 \) for some \( h_0 \) in \( B \). Hence \( i \), the identity on \( B \) and \( g \) the function defined by \( g(x) = h_0 \) for \( x \) in \( B \) are homotopic, a contradiction. Thus \( \theta \) is onto and the first part of the lemma is true.

If \( fj = hk \) for \( f \) and \( h \) in \( B \), \( j \) and \( k \) in \( T \), then \( e(fj) = e(hk) \). Since \( e \) is the identity for \( T \), it follows that \( j = ej = (ef)j = e(fj) = e(hk) = (eh)k = ek = k \).

**Lemma 11.** For \( f \) and \( h \) in \( B \), \( j \) and \( k \) in \( T \), \( (fj)(hk) = f(jk) \).

**Proof.** This follows easily from the fact that \( e \) is the identity for \( T \) and \( eg = e \) for any \( g \) in \( B \). For \( (fj)(hk) = f(je)(hk) = (fj)(eh)k = (fj)(ek) = f(jk) \).

With this lemma the proof of Theorem A is concluded.
Examples. (1) Let $S$ be the two-cell and define $m(x, y) = x|y|$, where $m$ is the multiplication on $S$ and the right side of the equality denotes multiplication of $x$ by the absolute value of $y$ in the ordinary way. In this example, the representation of $x = ft$, for $f$ in $B$ and $t$ in $T$ is unique. The following example shows that this need not always be true.

(2) Let $[e, f]$, $e \neq f$, be a line segment with multiplication for $x$ and $y$ in $[e, f]$ defined by $xy = x$. Let $L = [e, f] \times [0, 1]$, and for $(x, t)$ and $(y, s)$ let the product $(x, t)(y, s) = (x, ts)$ where the product $ts$ is real multiplication in the unit interval.

In $L [e, f] \times 0$ is a closed ideal, so denote by $Q$ the Rees quotient of $L$ with this ideal and let $\pi$ be the natural map from $L$ onto $Q$. From [7] $Q$ is a semigroup. Choose $a$ in $(0, 1)$ and consider $a$ fixed in the following discussion. Define $\theta: Q \rightarrow Q$ by

$$\theta(\pi(h, t)) = \begin{cases} 
\pi(e, t) & \text{for } h = e \text{ or } h = f \text{ and } t \in [0, a], \\
\pi(h, t) & \text{otherwise}.
\end{cases}$$

Let $Q_0 = \theta(Q)$. By defining multiplication in $Q_0$ in the obvious way, $Q_0$ is a semigroup.

Now let $L_1 = [e_1, f_1] \times [0, 1]$ be another semigroup as defined above, assuming for $x$ and $y$ in $[e_1, f_1]$, $xy = x$. The set $[e_1, f_1] \times [0, a]$ is a closed ideal of $L_1$. Denote by $P$ the Rees quotient of $L_1$ with this closed ideal and let $\pi_1$ be the natural map from $L_1$ to $P$.

Define $\delta: P \cup Q_0 \rightarrow P \cup Q_0$ in the following way:

$$\delta(s) = \begin{cases} 
\pi(e, t) & \text{for } s = \pi_1(e_1, t) \text{ where } t \in (a, 1], \\
\pi(f, t) & \text{for } s = \pi_1(f_1, t) \text{ where } t \in (a, 1], \\
\pi(e, a) & \text{for } s = \pi_1(e_1, t) \text{ where } t \in [0, a], \\
\pi(e, a) & \text{for } s = \pi_1(f_1, t) \text{ where } t \in [0, a], \\
s & \text{otherwise}.
\end{cases}$$

Let $S = \delta(P \cup Q_0)$. Clearly $S$ is topologically the two-cell. To define multiplication in $S$ it is necessary to consider only an element from $\delta(P \setminus \pi_1(h \times [0, 1]))$, where $h = e_1$ or $f_1$, and an element from $Q_0$. This is true since any pair of elements from other sets in $S$ already have their product defined. For $p = \pi_1(h, t)$ for some $h \in (e_1, f_1)$ and $s = \theta(\pi(g, r))$ for $g \in [e, f]$, $r$ and $t$ in $[0, 1]$, let $sp = \pi(g, rt)$ and $ps = \pi_1(h, rt)$. It is easily seen that multiplication defined in this way is continuous, associative and that for $x$ and $y$ in the boundary $xy = x$.

For the particular element $p = \pi(e, a)$, $p$ can be written as...
\( p = (\pi(e, 1))(\pi(e, a)) \) and also as \( p = (\pi(f, 1))(\pi(e, a)) \) and the idempotents \( \pi(e, 1) \) and \( \pi(f, 1) \) are distinct elements. In fact, for any idempotent in \([e_1, f_1]\), \( p = (\pi_1(h, 1))(\pi(e, a)) \).

**References**