ON MONOTONE AND POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. Second order nonlinear differential equations of the type

\[ y'' + f(y, y')y' + g(y) = 0 \quad (y' = dy/dx) \]

were first investigated by Levinson and Smith [4], and more recently by Utz [5; 6; 7; 8], Antosiewicz [1], G. Brauer [2] and others. For a detailed treatment see Lefschetz [3]. The purpose of this note is to consider the behavior of solutions of (1.1) for large values of the independent variable, assuming the existence of such solutions. A solution is said to be oscillatory if it has positive maxima and negative minima for arbitrarily large values of \( x \).

2. Monotone solutions. The following result gives sufficient conditions for monotone solutions of (1.1)

**Theorem 1.** Let the following conditions be satisfied:

(i) \( \phi(y, y') = f(y, y')y', f(y, y') \geq 0 \) for all \( y, y' \),

(ii) \( yg(y) \geq 0 \) for \( y' \neq 0 \),

(iii) \( \int_{0}^{y} g(z) dz \to -\infty \) as \( y \to -\infty \),

(iv) \( \int_{E} \left\{ \phi(y, y') + g(y) \right\} dy' \neq -\infty \),

where \( E = \{ x | y(x) \geq 0, y'(x) \geq 0 \} \), then a solution \( y(x) \), valid for all large \( x \), approaches zero monotonically as \( x \) tends to infinity.

H. A. Antosiewicz [1] has shown that (i), (ii) and (iii) imply that \( |y(x)| \) and \( |y'(x)| \) remain bounded as \( x \to \infty \). In [5] W. R. Utz established that (i), (ii), and (iii) also imply that if \( y(x) \) is a solution of (1.1) which does not vanish identically, then \( y(x) \) is bounded and oscillatory as \( x \to \infty \), or \( y(x) \) approaches zero monotonically as \( x \to \infty \). Let us assume that a solution \( y(x) \) of (1.1) does not approach zero monotonically as \( x \to \infty \); then \( y(x) \) must be oscillatory.

Let \( x_1 \) be a zero of \( y(x) \) such that \( y \) is negative immediately to the left and positive immediately to the right of \( x_1 \). Then \( y'(x_1) \geq 0 \). Moreover, if \( y'(x_1) = 0 \), then \( y'(x) > 0 \) on the immediate right of \( x_1 \) and consequently, \( y''(x) > 0 \) on the immediate right of \( x_1 \). But this is impossible as (1.1) shows. Hence \( y'(x_1) > 0 \). Let \( \tilde{x}_1 \) denote the first zero of \( y'(x) \) to the right of \( x_1 \). By integrating (1.1) over the interval \( (x_1, \tilde{x}_1) \) we have

Received by the editors September 2, 1958 and, in revised form, October 31, 1958.
(2.1) \[ y'(\tilde{x}_1) - y'(x_1) + \int_{x_1}^{\tilde{x}_1} [\phi(y, y') + g(y)] \, dx = 0. \]

Since \( y'(\tilde{x}_1) = 0 \),

(2.2) \[ y'(x_1) = \int_{x_1}^{\tilde{x}_1} [\phi(y, y') + g(y)] \, dx. \]

By (ii) \( g(y) > 0 \) for \( x_1 < x < \tilde{x}_1 \). Also \( \phi(y, y') \geq 0 \), and hence \( y''(x) < 0 \). Hence \( y' \) is positive and decreasing on \( (x_1, \tilde{x}_1) \). Thus, by (2.2) we have

\[ 1 = \int_{x_1}^{\tilde{x}_1} [\phi(y, y') + g(y)] \, dx/y'(x_1) \]

(2.3)

\[ \leq \int_{x_1}^{\tilde{x}_1} \left\{ [\phi(y, y') + g(y)]/y'(x) \right\} \, dx. \]

It follows from (iv) that the right-hand side of (2.3) tends to zero as \( x_1 \) tends to infinity. Hence, we arrive at a contradiction and \( y(x) \) is monotone decreasing to zero as \( x \to \infty \).

3. On positive solutions. The following theorem gives sufficient conditions for the nonexistence of solutions which remain positive for large values of \( x \).

**Theorem 2.** If (i) and (ii) of Theorem 1 hold, and

(v) \[ |\phi(y, y')| \leq g(y) \text{ for all } y, y', \]

(vi) there exist numbers \( a \) and \( b \) such that \( a < b \) and

\[ \int_b^{\infty} (x - a)[\phi(y, y') + g(y)] \, dx = \infty, \]

(vii) \[ (\phi(y, y') + g(y))^{1/2} \geq [y(x)]^2 \]

then no solution of (1.1) can remain positive for all \( x \) greater than \( a \).

Suppose that there exists a solution \( y(x) \) such that \( y(x) > 0 \) for \( x > a \). It follows from (1.1) that \( y''(x) < 0 \), so \( y'(x) \) is decreasing for \( x > a \), and \( y'(x) \) tends either to a finite limit or to \( -\infty \). Moreover this limit cannot be negative for then \( y(x) \) would become negative. Hence \( y(x) \) must be ultimately nondecreasing, and \( y'(x) \) tends to a finite non-negative limit. If we take \( x > a \) and integrate (1.1) over the interval \( (x, \infty) \), we obtain:

(3.1) \[ y'(\infty) - y'(x) + \int_x^{\infty} [\phi(y, y') + g(y)] \, dx = 0. \]

(The convergence of the integral is guaranteed by the existence of
\[ y'(\infty) = \lim_{x \to \infty} y(x). \] Since \( y'(\infty) \geq 0 \), we have, by (3.1)

\[ (3.2) \quad y'(x) \geq \int_x^\infty [\phi(y, y') + g(y)] dx. \]

Integrating (3.2) over the interval \((a, x)\) for \(x > a\) we have

\[ (3.3) \quad y(x) - y(a) \geq \int_a^x \int_u^x [\phi(y, y') + g(y)] dt du \]

and thus

\[ (3.4) \quad y(x) \geq \int_a^x (t - a)[\phi(y, y') + g(y)] dt. \]

From (3.4) and by making use of (vii), we have the following inequalities,

\[ \frac{(x - a)[\phi(y, y') + g(y)]^{1/2}}{\left\{ \int_a^x (t - a)[\phi(y, y') + g(y)] dt \right\}^2} \geq \frac{[y(x)]^2(x - a)}{\left\{ \int_a^x (t - a)[\phi(y, y') + g(y)] dt \right\}^2} \geq x - a. \]  

If we multiply both sides of (3.5) by \([\phi(y, y') + g(y)]^{1/2}\), choose \(x_1\) such that \(a < x_1\), and integrate (3.5) from \(b\) to \(x_1\) we have

\[ (3.6) \quad \int_b^{x_1} \left\{ \int_a^x (t - a)[\phi(y, y') + g(y)] dt \right\}^{-2} \left\{ (x - a)[\phi(y, y') + g(y)] \right\} dx \]

\[ \geq \int_b^{x_1} (t - a)[\phi(y, y') + g(y)]^{1/2} dt. \]

As \(x_1 \to \infty\) the left-hand side of (3.6) remains bounded, hence we reach a contradiction to (vi). Thus \(y(x)\) cannot remain positive for all large values of \(x\).

Referee’s Remark. The author’s proof of Theorem 2 also shows that \(y(x)\) cannot be negative for all large values of \(x\). Thus \(y(x)\) changes sign at arbitrarily large \(x\)-values, that is, \(y\) is oscillatory.

References


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