ON AUTOMORPHIC-INVERSE PROPERTIES IN LOOPS

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Introduction. In a loop $(G, \cdot)$ we define $J$ as the mapping that takes every element into its right inverse, i.e., $x \cdot xJ = 1$, for all $x$ in $G$. It is well known \[3\] that in I.P. loops $J$ is an anti-automorphism. In crossed-inverse loops (in short C.I. loops), which are defined as satisfying either of the equivalent identities $xy \cdot xJ = y$ or $x(y \cdot xJ) = y$, for all $x$ and $y$ in $G$, $J$ is known \[1\] to be an automorphism. On the other hand, easily constructed counter-examples show that a loop in which $J$ is an anti-automorphism is not necessarily I.P., and that a loop in which $J$ is an automorphism is not necessarily C.I. Furthermore a loop which is I.P. everywhere, i.e. all of whose isotopes are I.P., is known \[4\] to be Moufang. A loop which is C.I. everywhere is an abelian group; this can be easily checked by computation, but it is also obvious, due to the fact that the C.I. property can be represented in a 3-web by the validity of the Thomsen figure (cf. \[7\]) in a special position while the general Thomsen figure corresponds to both associativity and commutativity.

It is the purpose of this paper to show that an I.P. loop in which $J$ is an anti-automorphism everywhere is Moufang, and that a C.I. loop in which $J$ is an automorphism everywhere is an abelian group. Thus we show that the weaker “automorphic-inverse properties” $(xy)J = yJ \cdot xJ$ and $(xy)J = xJ \cdot yJ$ in I.P. loops and C.I. loops, respectively, are sufficient to replace the full I.P. and C.I. properties of the isotopes.

Theorem 1. An I.P. loop, in all of whose isotopes $J$ is an anti-automorphism, is Moufang.

Proof. We consider an isotope with $xg \cdot y = xy$. Its unit is $g$. Let $xJ^*$ be the right inverse of $x$ in the isotope. Using the inverse property, which implies $J = J^{-1}$, we have then

$$gx^{-1} \cdot xJ^* = g,$$

and

$$J^* = JL(g)R(g).$$

The theorem assumes $(x \cdot y)J^* = yJ^* \cdot xJ^*$, that is $g(xg^{-1} \cdot y)^{-1} \cdot g = ([gy^{-1} \cdot g]g^{-1})(gx^{-1} \cdot g)$. With $gx^{-1} = a, y^{-1} = b$ we get $(g \cdot ba)g = gb \cdot ag$, for all $a, b, g$ in the loop. Thus the loop is Moufang.

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Theorem 2. A C.I. loop, $G$, in all of whose isotopes $J$ is an automorphism, is an abelian group.

Proof. The proof will be performed in 5 stages, (i) to (v).

(i) The inverses in $G$ are unique.

Let the isotope be defined by $xg * fy = xy$. Then we have, using the C.I. property,

$$ (gJ^{-1} * x)(xJ* * fJ) = fg, \quad \text{and} \quad J* = JL(g)L(fg)L(f). $$

The identity $(x * y)J* = xJ* * yJ*$ becomes

$$ f[gJ^{-1} * (g * fJ)(gJ^{-1} * (g * fJ)))] = [gJ^{-1} * (f(fJ * gJ^{-1}))][fJ * (g * fJ)]. $$

Putting $x = g^2g$ and $y = f = 1$, yields $1 = (gJ^{-1} * gJ)g^2$, and finally $gJ^{-1} = gJ$. Thus inverses are unique in $G$.

(ii) The squares form a normal subloop and lie in the centre.

$$ f[gJ^{-1} * (g * fJ)(gJ^{-1} * (g * fJ)))] = [gJ^{-1} * (f(fJ * gJ^{-1}))][fJ * (g * fJ)]. $$

Putting $x = g^2g$ and $y = f = 1$, yields $1 = (gJ^{-1} * gJ)g^2$, and finally $gJ^{-1} = gJ$. Thus inverses are unique in $G$.

If we substitute in (1) $gx^{-1} = a$, $y^{-1}f = b$ we get

$$ f[gJ^{-1} * (g * fJ)(gJ^{-1} * (g * fJ)))] = [gJ^{-1} * (f(fJ * gJ^{-1}))][fJ * (g * fJ)]. $$

This is an autotopism (cf. [4])

$$ L(fg)L(f)L(g^{-1}), L(f^{-1}L(g)L(fg), L(g)L(fg)L(f)). $$

With $f = 1$ this becomes

$$ (L(g)L(g^{-1}), L(g)L(g), L(g)L(g)). $$

According to [2] every autotopism $(U, V, W)$ in a C.I. loop implies another autotopism $(L(1U), L(1V), L(1W))$. Thus (3) implies $(I, L(g^2), L(g^2))$ as a new autotopism, that is,

$$ a * g^2b = g^2 * ab. $$

With $b = 1$ this yields $ag^2 = g^2a$, hence the squares of the elements commute with all elements of $G$. Equation (4) becomes $a * bg^2 = ab * g^2$. Thus all the squares belong to the right nucleus, but, as the author has proved [2], all the elements of the right nucleus are centre elements, and therefore all squares of loop elements lie in the centre. Moreover the squares form a normal subloop:

$$ x^2y^2 = x^2y^2(xy * x^{-1}y^{-1}) = xy * (x^2x^{-1} * y^2y^{-1}) = xy * xy = (xy)^2, $$

and $(x^2)J = (xJ)^2$.

(iii) $G$ is commutative.

According to [2], the elements $1U$ and $1V$, for all autotopisms $(U, V, W)$ of $G$, form a Moufang subloop, and the cubes of the elements of this subloop lie in the centre of $G$. Since also the squares lie in the centre, $1U$ and $1V$ themselves lie in the centre. In particular
take as \((U, V, W)\) the autotopism (2), then \(1U = g^{-1}(f \cdot fg)\) and \(1V = fg \cdot gf^{-1}\) lie in the centre of \(G\). Therefore we have
\[
\begin{align*}
g^{-1}(g^{-1}f \cdot 1V) &= (g^{-1} \cdot g^{-1}f) \cdot 1V, \\
f &= g^{-1}(g^{-1}f \cdot (fg \cdot gf^{-1})) = (g^{-1} \cdot g^{-1}f)(fg \cdot gf^{-1}), \\
(5) \quad f(g \cdot gf^{-1}) &= fg \cdot gf^{-1}
\end{align*}
\]
and
\[
\begin{align*}
1V \cdot f^{-1}g^{-1} &= f^{-1}g^{-1} \cdot 1V, \\
gf^{-1} &= (fg \cdot gf^{-1}) \cdot f^{-1}g^{-1} = f^{-1}g^{-1} \cdot (fg \cdot gf^{-1}), \\
(6) \quad gf^{-1} \cdot fg &= fg \cdot gf^{-1}.
\end{align*}
\]
We interchange \(f\) and \(g\) in (5) and get
\[
\begin{align*}
g(f \cdot fg^{-1}) &= gf \cdot fg^{-1} = (gf^{-1} \cdot fg) f^2g^{-2}, \\
\end{align*}
\]
In view of (6) this becomes
\[
\begin{align*}
g^{-1}(f \cdot fg) &= (f \cdot gf^{-1}) f^2g^{-2}, \text{ or } 1U = 1V \cdot f^2g^{-2}.
\end{align*}
\]
Now obviously \(1W = 1U \cdot 1V\), hence \(1W = 1V \cdot f^2g^{-2} \cdot 1V\), that is, \(f(fg \cdot g) = f^2g^{-2}(fg \cdot gf^{-1})^2\), and using the multiplication rule for squares, \(f(fg \cdot g) = f^2g^{-2}f^2g^2gf^2f^{-2} = f^2g^2\), and \(fg = f^2g \cdot g^{-1}f^{-1} = gf\). Thus the commutativity of \(G\) is established.

The commutativity and the C.I. property imply the inverse property. Moreover the alternative property holds:
\[
a \cdot ab = a^2(a \cdot a^{-1}b) = a^2b.
\]

(iv) \(G\) is an A-loop.

Using the associativity of multiplication by squares, we may write the autotopism (2)
\[
(U, V, W) = (L(fg)^{-1}L(f)L(g), L(f)L(g)L(fg)^{-1}, L(g)L(fg)^{-1}L(f)).
\]
Now \(1U = 1V = 1\), and that implies \(U = V = W\). Hence \(V = L(f)L(g)L(fg)^{-1}\) is an automorphism. We shall prove now that every inner mapping of \(G\) is the product of inner mappings of the form \(L(f)L(g)L(fg)^{-1}\). Owing to the commutativity of \(G\) and the property \(L^{-1}(x) = L(x^{-1})\) of I.P. loops, every inner mapping \(S\) of \(G\) has the form \(S = \prod_{k=1}^{n} L(a_k), \) with \(1S = 1\). We can write
\[
S = \left[ L(a_1)L(a_2)L(a_1a_2)^{-1} \right] \left[ L(a_1a_2)L(a_3)L(a_1a_2^{-1}a_3)^{-1} \right] \cdots \left[ L \left( 1 \prod_{k=1}^{n-1} L(a_k) \right) L(a_n)L \left( 1 \prod_{k=1}^{n} L(a_k) \right)^{-1} \right]
\]
because the last term of each bracket and the first term of the next
bracket are of the type \( L(x)L(x^{-1}) \) and cancel out; furthermore the last term of the last bracket is

\[
L \left( 1 \prod_{k=1}^{n} L(a_k) \right)^{-1} = L(1S)^{-1} = L(1^{-1}) = I.
\]

Each of the brackets contains a product of the form

\[
V = L(f)L(g)L(fg)^{-1}
\]

which is an automorphism. Hence every inner mapping \( S \) is a product of automorphisms, and therefore an automorphism. A loop in which every inner mapping is an automorphism is usually called an \( A \)-loop.

(v) \( G \) is abelian.

\( G \) is a commutative I.P. loop. \( A \)-loops with the inverse property are diassociative \([5]\), and if they are also commutative they are Moufang \([6]\). Thus we have the identity \( (xy \cdot x)z = x(y \cdot xz) \), and therefore \( x^2y \cdot z = x(y \cdot xz) \), \( yz = x^{-1}(y \cdot xz) \), and \( yz \cdot x = y \cdot xz \). Hence \( G \) is associative and therefore abelian.

**Remark.** In recent work (to appear in Pacific J. Math.) J. M. Osborn has been dealing with a generalization of both C.I. and I.P. loops: “weak-inverse loops” in which \( xy \cdot z = 1 \) if, and only if, \( x \cdot yz = 1 \). In weak-inverse loops \( J^2 \) is an automorphism, but this property is not sufficient for defining weak-inverse loops. Osborn has investigated loops which are weak-inverse everywhere, and these loops have interesting properties. In connection with the present paper it might be of interest whether a weak-inverse loop all of whose isotopes have \( J^2 \) as an automorphism (and are not necessarily weak-inverse) has already the same properties, such that again an “automorphic-inverse property” would suffice to replace the full loop identity in the isotopes.

**References**


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