A RADICAL ALGEBRA WITHOUT DERIVATIONS
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Wermer and Singer [1] have shown that in a semi-simple commutative Banach algebra there exist no nontrivial derivations, a derivation being a bounded linear operator $D$, taking the algebra into itself, with the additional property that

$$D(u \cdot v) = u \cdot (Dv) + (Du) \cdot v.$$

Wermer has conjectured the following converse: If a commutative Banach algebra has no nontrivial derivations then it is semi-simple.

A weaker statement is: If a commutative Banach algebra is all radical [i.e. $x^n \to 0$ for all $x$] then it has a nontrivial derivation.

In this note we show that even this weaker statement is false.

We choose a fixed sequence $\lambda_n$, $n = 1, 2, \cdots$, of non-0 complex numbers, and consider the following algebraic system $S$.

The elements are those formal power series

$$a(t) = \sum_{n=1}^{\infty} a_n t^n$$

for which $\sum |a_n| |\lambda_n|^n < \infty$.

Addition, multiplication, and multiplication by scalars is as usual.

Consider now the following properties

A: $|\lambda_n| \geq |\lambda_{n+1}|$, $n = 1, 2, \cdots$,
B: $\lambda_n \to 0$,
C: $n^\epsilon |\lambda_{n+1}|^{n+1}/|\lambda_n|^n \to \infty$ for any fixed $\epsilon > 0$.

We prove the following lemmas:

**Lemma 1.** A $\Rightarrow S$ is a Banach algebra under the norm $||a|| = \sum |a_n| |\lambda_n|^n$.

**Lemma 2.** A and B $\Rightarrow S$ is all radical.

**Lemma 3.** A and C $\Rightarrow S$ has no nontrivial derivations.

**Proof 1.** Since $S$, under this norm is isomorphic and isometric to $l^1$ under the correspondence

$$\sum_{n=1}^{\infty} a_n t^n \leftrightarrow \{ a_n(\lambda_n)^n \},$$

$S$ is clearly a Banach space. Also

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584
\[ \|ab\| = \left\| \sum_{n=2}^{\infty} \left( \sum_{m+k=n} a_m b_k \right) t^n \right\| \]
\[ = \sum_{n=2}^{\infty} \left\| \sum_{m+k=n} a_m b_k \right\| \cdot |\lambda_n|^n \]
\[ \leq \sum_{m,k} |a_m| \cdot |b_k| \cdot |\lambda_{m+k}|^{m+k} = \sum_{m,k} |a_m| \cdot |b_k| \cdot |\lambda_{m+k}|^m \cdot |\lambda_{m+k}|^k \]
\[ \leq \sum_{m,k} |a_m| \cdot |b_k| \cdot |\lambda_m|^m \cdot |\lambda_k|^k \quad \text{(by A.)} \]
\[ = \|a\| \cdot \|b\| \]

and so is a Banach algebra.

**Proof 2.** It is known that the radical \( R \) is a closed subalgebra. But now \( \|t^n\|^{1/n} = |\lambda_n| \) and by B this \( \rightarrow 0 \). \( \therefore t \in R \), by the algebraic closure of \( R \), \( P(t) \in R \), \( P \) any polynomial, \( \therefore \) by topological closure, all

\[ a(t) \in R : \left[ \|a(t) - (a_1 + \cdots + a_N t^N)\| \right. \]
\[ = \sum_{n=N+1}^{\infty} |a_n| \cdot |\lambda_n|^n \rightarrow 0 \quad \text{with } N. \]

**Proof 3.** Since, as in the above parenthetical remark, the polynomials are dense in \( S \) it suffices to prove that \( D(t) = 0 \) for \( D \) any derivation, for it would then follow that \( D(P(t)) = P'(t)Dt = 0 \) and so \( D(a(t)) = 0 \) or \( D \) is trivial.

Let

\[ D(t) = \sum_{m=1}^{\infty} c_m t^m. \]

Then

\[ D(t^n) = nt^{n-1}D(t) = n \sum_{k=0}^{\infty} c_{k+1} t^{k+n}. \]

\( \therefore \) for any fixed \( k = 0, 1, \cdots \)

\[ \|D(t^n)\| \geq n \cdot \left| c_{k+1} \right| \cdot \left| \lambda_{k+n} \right|^{k+n}, \]
on the other hand, \( D \) being bounded,

\[ \left| c_{k+1} \right| \leq M \left| \lambda_n \right|^n = \left| \lambda_{n+k} \right|^{n+k}. \]

Now as \( n \rightarrow \infty \) it follows from C that the right side \( \rightarrow 0 \), \( \therefore c_{k+1} = 0. \)
this holding for each $k = 0, 1, \cdots$ it follows that $D(t) = 0$.

We now see that a counterexample to Wermer’s conjecture is afforded us once we note that conditions A, B, C are not contradictory. This is clear, however, since e.g.

$$
\lambda_n = \frac{1}{\log(n + 1)}
$$

satisfies all three of them.

**Reference**