SOME CONDITIONS UNDER WHICH A HOMOGENEOUS CONTINUUM IS A SIMPLE CLOSED CURVE

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In a recent paper [3], the author added a note in proof stating that two of the results could be strengthened by using the fact that a non-degenerate continuous curve is a simple closed curve if it is nearly homogeneous and is not a triod. It is the main purpose of this note to present a proof of this theorem and to state stronger forms of two results in [3]. Also, a theorem is presented that is related to a question raised by Knaster and Kuratowski [6]. This question as to whether every nondegenerate homogeneous bounded plane continuum is a simple closed curve has been settled negatively with examples by Bing [1] and Bing and Jones [2]. Additional conditions under which there is an affirmative answer have been given in some of the references cited in [3]. It apparently has not been noticed previously that there is an affirmative answer for nondegenerate bounded continua that are homogeneously embedded in the plane. This result (Theorem 3) follows directly from a characterization of homogeneous decomposable bounded plane continua given by Jones [5] and the nonaccessibility of certain points of indecomposable plane continua [7].

Definitions. A continuous curve is a compact metric space that is connected and locally connected. A triod is a continuum which is separated into three mutually separated sets by one of its subcontinua. A continuum $M$ is: (i) homogeneous if for any two points $x$ and $y$ of $M$ there is a homeomorphism of $M$ onto itself that carries $x$ into $y$; (ii) nearly homogeneous if for any point $x$ of $M$ and any subset $D$ of $M$, open relative to $M$, there is a homeomorphism of $M$ onto itself that carries $x$ into a point of $D$; (iii) 2-homogeneous if for any two points $x_1$ and $x_2$ of $M$ and any two points $y_1$ and $y_2$ of $M$ there is a homeomorphism of $M$ onto itself that carries $x_1+x_2$ onto $y_1+y_2$; (iv) nearly 2-homogeneous if for any two points $x_1$ and $x_2$ of $M$ and any two subsets $D_1$ and $D_2$ of $M$ that are open relative to $M$ there exist two points $y_1$ and $y_2$ in $D_1$ and $D_2$, respectively, and a homeomorphism of $M$ onto itself that carries $x_1+x_2$ onto $y_1+y_2$; (v) homogeneously embedded in a space $S$ if for any two points $x$ and $y$ of $M$ there is a homeomorphism of $S$ onto itself that carries $x$ into $y$ and $M$ onto itself.

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Theorem 1. If the nondegenerate continuous curve $M$ is nearly homogeneous and is not a triod, then $M$ is a simple closed curve.

Proof. There exist a closed subset $K$ of $M$ and two points $p_1$ and $p_2$ of $M$ such that $K$ is irreducible with respect to the property of separating $p_1$ from $p_2$ in $M$ and $M - K$ is the sum of two mutually separated sets $M_1$ and $M_2$ that contain $p_1$ and $p_2$, respectively. Let $G_1, G_2, G_3, \ldots$ be a sequence of finite collections of connected open subsets of $M$ such that for each $i$, (1) $G_i$ covers $K$, (2) each element of $G_i$ intersects $K$ and has a diameter less than $1/i$, and (3) each element of $G_{i+1}$ is a subset of some element of $G_i$. There is a finite collection $T_1$ of arcs such that (1) $T_1^*$ is connected, (2) each arc of $T_1$ intersects an element of $G_1$, and (3) each element of $G_1$ intersects an arc of $T_1$. Now define a sequence $T_1, T_2, T_3, \ldots$ of finite collections of arcs such that for each $i \ (i > 1)$, (1) each arc of $T_i$ is a subset of an element of $G_{i-1}$ and intersects an arc of $T_{i-1}$, (2) each arc of $T_i$ intersects an element of $G_i$, and (3) each element of $G_i$ intersects an arc of $T_i$. Let $K'$ denote the continuum $K + T_1^* + T_2^* + \ldots$.

Suppose that $M$ is not a simple closed curve. That $K'$ contains neither $M_1$ nor $M_2$ follows from the fact that a compact metric continuum is a simple closed curve provided it is nearly homogeneous and some arc in it contains a set that is open relative to that continuum. Hence $M - K'$ is the sum of two mutually separated sets $M_1'$ and $M_2'$, where $M_1' = M_1 - K' \ (i = 1, 2)$. Since $K' - K$ is nowhere dense in $M$ and each point of $K$ is a limit point of both $M_1$ and $M_2$, it follows that each point of $K$ is a limit point of both $M_1'$ and $M_2'$.

Now from the near-homogeneity of $M$, it follows that there is a homeomorphism $f$ of $M$ onto itself that carries some point $x$ of $K$ into $M_1'$. Then the point $f(x)$ is a limit point of both $f(M_1')$ and $f(M_2')$, so that $M_1'$ intersects both $f(M_1')$ and $f(M_2')$. There exists an arc $H$ in $M$ such that $H + K' + f(K')$ is a continuum that is nowhere dense in $M$. Let $N = H + K' + f(K')$. Then $M - N$ is the sum of the three mutually separated sets $M_2' - N, M_1' \cdot f(M_1') - N$, and $M_1' \cdot f(M_2') - N$, and this is contrary to the hypothesis that $M$ is not a triod. Hence $M$ is a simple closed curve.

Question. If the continuous curve $M$ has no local separating point and $p_1$ and $p_2$ are two points of $M$, then does there exist a subcontinuum $K$ of $M$ such that (1) $M - K$ is the sum of two mutually

\footnote{If $L$ is a collection of point sets, then $L^*$ denotes the set which is the sum of the elements of $L$.}

\footnote{This method of constructing $K'$ is similar to a method used by Zippin [8], but his result is not directly applicable here.}
separated sets \( M_1 \) and \( M_2 \) containing \( p_1 \) and \( p_2 \), respectively, and (2) every point of \( K \) is a limit point of both \( M_1 \) and \( M_2 \)?

**Theorem 2.** If the decomposable compact metric continuum \( M \) is nearly 2-homogeneous and is not a triod, then \( M \) is a simple closed curve.

**Proof.** It follows from Theorem 15 of [3] that \( M \) is a continuous curve. Since \( M \) is nearly homogeneous, it follows from Theorem 1 that \( M \) is a simple closed curve.

**Corollary.** If the nondegenerate compact metric continuum \( M \) is 2-homogeneous and is not a triod, then \( M \) is a simple closed curve.

**Theorem 3.** If the nondegenerate bounded continuum \( M \) is homogeneously embedded in a plane \( E \), then \( M \) is a simple closed curve.

**Proof.** Suppose that \( M \) is not a simple closed curve. It follows from two results by F. B. Jones [4; 5] that some nondegenerate subcontinuum \( K \) of \( M \) is indecomposable. Mazurkiewicz [7] has shown that some point \( y \) of \( K \) is not accessible from the complement of \( K \), and hence \( y \) is not accessible from the complement of \( M \). Since some point \( x \) of \( M \) is accessible from the complement of \( M \), this leads to the contradiction that there is no homeomorphism of \( E \) onto itself that carries \( x \) into \( y \) and \( M \) onto itself.

**Bibliography**


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