A TAUBERIAN THEOREM FOR STRONG RIESZ SUMMABILITY

PRAMILA SRIVASTAVA

1.1. Let $\sum_{n=1}^{\infty} a_n$ be a given infinite series, and λ_n a positive monotonic increasing function of n, tending to infinity with n. For $r > -1$, integral or not, we write

$$A_\lambda^r(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^r a_n.$$

When $k \geqq 0$, the series $\sum a_n$ is said to be summable (R, λ, k) to the value s, if

$$\lim_{x \to \infty} x^{-k} A_\lambda^k(x) = s$$

[3], see also [2]. The series $\sum a_n$ is said to be strongly summable (R, λ, k), or summable $[R, \lambda, k]$, where $k > 0$, if

$$\int_{\lambda_0}^{\omega} \left| x^{-(k-1)} A_\lambda^{(k-1)}(x) - s \right| dx = o(\omega),$$

as $\omega \to \infty$. If

$$\sum_{\lambda_n < \omega} \left| a_n \lambda_n \right| = o(\omega),$$

and $\sum a_n$ is convergent, then it is said to be summable $[R, \lambda, 0]$. Replacing o by O and convergence by boundedness in the above, the definitions for boundedness $[R, \lambda, k]$ and $[R, \lambda, 0]$ are obtained [4].

1.2. The first theorem of consistency for strong Riesz summability [4, Theorems 3 and 3′] states that if $r \geqq 0$ and the infinite series $\sum a_n$ is summable $[R, \lambda, r]$, then it is summable $[R, \lambda, k]$ when $k > r$. It is trivial that a series summable $[R, \lambda, k]$ may fail to be summable $[R, \lambda, r]$ when $r < k$. A result has already been obtained in the direction of determining some extra condition or conditions which together with summability $[R, \lambda, k]$ may yield summability $[R, \lambda, r]$, r being less than k [4, Theorem 8]. The object of this paper is to establish another theorem with reference to the same problem.

It may be mentioned that the theorem of the present paper includes as a particular case a theorem of Winn [5, Theorem X] for

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[1] The condition $\mu \leqq k$ of the original lemma is superfluous in view of the extension of the definition for $A_\lambda^k(\omega)$ to negative k, $k > -1$.

strong Cesàro summability on account of equivalence of summabilities $[R, n, k]$ and $[C, k]$ [**4**; **1**].

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2.1. THEOREM. *If* $\sum a_n$ *is summable* (R, λ, k) *for some positive* k *and is bounded* $[R, \lambda, r]$ *for some* $r \geqq 0$, *then it is summable* $[R, \lambda, r+\delta]$ *whenever* $\delta > 0$.

2.2. In order to prove the theorem we require the following lemmas.

LEMMA 1 [**3**, p. 27, Lemma 6].[1] *If* $k > 0$, $\mu < 1$, *then*

$$A_\lambda^{k-\mu}(\omega) = \frac{\Gamma(k-\mu+1)}{\Gamma(k+1)\Gamma(1-\mu)} \int_0^\omega A_\lambda^{k-1}(u)(\omega-u)^{-\mu} du.$$

LEMMA 2 [**4**, Theorem 1]. *If* $\sum a_n$ *is summable* (*or bounded*) $(R, \lambda, k-1)$, *then it is summable* (*or bounded*) $[R, \lambda, k]$.

LEMMA 3. *If* $\sum a_n$ *is summable* $[R, \lambda, k+1]$ *and bounded* $[R, \lambda, k]$, *then it is summable* $[R, \lambda, k+\delta]$ *when* $k > 0$ *and* $\delta > 0$.

PROOF OF LEMMA 3. We suppose $\delta < 1$, as otherwise the result is an immediate consequence of the first theorem of consistency for strong Riesz summability. We may, also, suppose without any loss of generality the sum of the given series to be zero. Hence, being given that

$$\int_{\lambda_0}^\omega \left| x^{-(k-1)} A_\lambda^{k-1}(x) \right| dx = O(\omega) \tag{2.1}$$

and

$$\int_{\lambda_0}^\omega \left| x^{-k} A_\lambda^{k}(x) \right| dx = o(\omega), \tag{2.2}$$

it is to be shown that

$$\int_{\lambda_0}^\omega \left| x^{-(k+\delta-1)} A_\lambda^{k+\delta-1}(x) \right| dx = o(\omega). \tag{2.3}$$

The conditions (2.1), (2.2) and (2.3) may be replaced by the following alternative forms respectively.

$$\int_{\lambda_0}^\omega \left| A_\lambda^{k-1}(x) \right| dx = O(\omega^k), \tag{2.1'}$$

$$\int_{\lambda_0}^{\omega} | A_\lambda^k(x) | dx = o(\omega^{k+1}), \tag{2.2'}$$

$$\int_{\lambda_0}^{\omega} | A_\lambda^{k+\delta-1}(x) | dx = o(\omega^{k+\delta}). \tag{2.3'}$$

We have, by Lemma 1,

$$\begin{aligned} C\int_{\lambda_0}^{\omega} | A_\lambda^{k+\delta-1}(x) | dx &= \int_{\lambda_0}^{\omega} \left| \int_0^x A_\lambda^{k-1}(u)(x-u)^{\delta-1} du \right| dx \\ &\leqq \int_{\lambda_0}^{\omega} dx \left| \int_0^{x-t} (x-u)^{\delta-1} A_\lambda^{k-1}(u) du \right| \\ &\quad + \int_{\lambda_0}^{\omega} dx \left| \int_{x-t}^{x} (x-u)^{\delta-1} A_\lambda^{k-1}(u) du \right| \\ &= I_1 + I_2, \end{aligned}$$

say, where C is a constant which may be different at different places. Here $t=\xi x$ and ξ is a constant to be determined subsequently. Evidently, it will suffice to show that

$$I_1 = o(\omega^{k+\delta})$$

and

$$I_2 = o(\omega^{k+\delta}).$$

We first consider I_2. We have

$$\begin{aligned} I_2 &= \int_{\lambda_0}^{\omega} dx \left| \int_{x-t}^{x} (x-u)^{\delta-1} A_\lambda^{k-1}(u) du \right| \\ &\leqq \int_{\lambda_0}^{\omega} dx \int_0^t u^{\delta-1} | A_\lambda^{k-1}(x-u) | du \\ &= \int_{\lambda_0}^{\omega} dx \int_0^{\xi x} u^{\delta-1} | A_\lambda^{k-1}(x-u) | du. \end{aligned}$$

By changing the order of integration, we get

$$\begin{aligned} I_2 &\leqq \int_0^{\lambda_0\xi} u^{\delta-1} du \int_{\lambda_0}^{\omega} | A_\lambda^{k-1}(x-u) | dx \\ &\quad + \int_{\lambda_0\xi}^{\omega\xi} u^{\delta-1} du \int_{u/\xi}^{\omega} | A_\lambda^{k-1}(x-u) | dx \\ &= I_{2,1} + I_{2,2}, \end{aligned}$$

say. Now, by (2.1′)

$$I_{2,1} = O\int_0^{\lambda_0\xi} u^{\delta-1}(\omega - u)^k du$$
$$= O\{\omega^k(\lambda_0\xi)^\delta\} = o(\xi^\delta\omega^{k+\delta}).$$

And

$$I_{2,2} = O\int_{\lambda_0\xi}^{\omega\xi} u^{\delta-1}(\omega - u)^k du + O\int_{\lambda_0\xi}^{\omega\xi} u^{k+\delta-1}\left(1 - \frac{1}{\xi}\right)^k du$$
$$= O(\omega^{k+\delta}\xi^\delta) + O(\omega^{k+\delta}\xi^\delta) = O(\omega^{k+\delta}\xi^\delta).$$

Supposing $\xi = o(1)$, we obtain

$$I_2 \leqq I_{2,1} + I_{2,2}$$
$$= o(\omega^{k+\delta}),$$

as $\omega \to \infty$. Also

$$I_1 = \int_{\lambda_0}^{\omega} dx \left| \int_0^{x-t} (x - u)^{\delta-1} A_\lambda^{k-1}(u) du \right|$$
$$\leqq C\int_{\lambda_0}^{\omega} dx \mid A_\lambda^k(x - t) \mid t^{\delta-1} + C\int_{\lambda_0}^{\omega} dx \left| \int_0^{x-t} A_\lambda^k(u)(x - u)^{\delta-2} du \right|$$
$$= CI_{1,1} + CI_{1,2},$$

say. By (2.2′), we get

$$I_{1,1} = \int_{\lambda_0}^{\omega} (x\xi)^{\delta-1} \mid A_\lambda^k\{x(1 - \xi)\} \mid dx$$
$$= o(\omega\xi)^{\delta-1}\omega^{k+1}(1 - \xi)^k + o\left\{\xi^{\delta-1}(1 - \xi)^k \int_{\lambda_0}^{\omega} x^{\delta-2+k+1} dx\right\}$$
$$= o\{\omega^{k+\delta}\xi^{\delta-1}(1 - \xi)^k\} + o\{\xi^{\delta-1}(1 - \xi)^k\omega^{k+\delta}\}$$
$$= o\{\omega^{k+\delta}\xi^{\delta-1}(1 - \xi)^k\},$$

and

$$I_{1,2} \leqq \int_{\lambda_0}^{\omega} dx \int_0^{x-t} \mid A_\lambda^k(u) \mid (x - u)^{\delta-2} du$$
$$= \int_{\lambda_0}^{\omega} t^{\delta-2} o(x - t)^{k+1} dx$$
$$= \int_{\lambda_0}^{\omega} x^{\delta-2+k+1}\xi^{\delta-2}(1 - \xi)^{k+1} dx$$
$$= o\{\omega^{k+\delta}\xi^{\delta-2}(1 - \xi)^{k+1}\}.$$

Hence

$$I_1 = O(I_{1,1} + I_{1,2}) = o(\omega^{k+\delta}),$$

by choosing ξ first and then ω. Thus the proof of the lemma is complete.

2.3. **Proof of the theorem.** By Lemma 2, for the proof of the theorem it is sufficient to show simply that "if $r \geqq 0$ and $\sum a_n$ is bounded $[R, \lambda, r]$ and summable $[R, \lambda, k]$, then it is also summable $[R, \lambda, r+\delta]$ where $\delta > 0$." We consider first the case $r > 0$. If $k \leqq r+1$, the theorem is seen to follow immediately from Lemma 3. And, if $k > r+1$, by the first theorem of consistency $\sum a_n$ is bounded $[R, \lambda, k-1]$ and summability $[R, \lambda, k-1+\delta]$ follows from Lemma 3. By similar repeated applications of Lemma 3, ultimately, summability $[R, \lambda, r+\delta]$, $\delta > 0$, of the given series is obtained.

For the case $r=0$, we observe that given a finite number $\delta > 0$, we can always write $\delta = \delta_1 + \delta_2$, where both δ_1 and δ_2 are >0. Now, since the series $\sum a_n$ is bounded $[R, \lambda, 0]$, it is bounded $[R, \lambda, \delta_1]$ by the first theorem of consistency, and then summability $[R, \lambda, \delta_1+\delta_2]$ i.e. summability $[R, \lambda, \delta]$ follows from the part of the theorem already proved. This completes the proof of the theorem.

2.4. The reasoning, given above for the case $r=0$, leads to the following corollary.

COROLLARY. *If $\sum a_n$ is bounded $[R, \lambda, k+\delta]$ for every $\delta > 0$ and is summable (R, λ, l) for some l, then it is also summable $[R, \lambda, k+\delta]$.*

2.5. Finally it may be pointed out that we can not take δ to be zero in our theorem. A gegenbeispiel to this effect is provided by the well known alternating series $1-1+1- \cdots$.

This series is bounded $[R, n, 1]$, summable (R, n, k), $k > 0$, but not summable $[R, n, 1]$.

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ALLAHABAD UNIVERSITY, ALLAHABAD, INDIA