COEFFICIENTS IN CERTAIN ASYMPTOTIC FACTORIAL EXPANSIONS

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1. Introduction. Let \( p \) and \( q \) denote integers with \( 0 \leq p \leq q \), then it is known from [2; 3] and [7] that the function

\[
g(w) = \prod_{i=1}^{p} \Gamma(w + \sigma_i) / \prod_{j=0}^{q} \Gamma(w + \rho_j),
\]

where the \( \sigma_i \) and \( \rho_j \) are independent of \( w \), admits for large \( |w| \) in the sector \( |\arg w| < \pi - \epsilon \), an asymptotic expansion of the form

\[
g(w) = ((2\pi)^{1/2})^{1-\alpha} \alpha^{-\alpha w + \beta - 1/2} \left[ \sum_{m=0}^{M} c_m / \Gamma(\alpha w + \beta + m) \right.
\]

\[
+ O(1/\Gamma(\alpha w + \beta + M + 1)) \Big] .
\]

Here, and in the sequel, \( M \) denotes a non-negative integer independent of \( w \), \( \epsilon \) any positive constant and

\[
(1.3) \quad c_0 = 1; \quad \alpha = q + 1 - p; \quad \beta = \sum_{j=0}^{q} \rho_j - \sum_{i=1}^{p} \sigma_i + (1 - \alpha)/2.
\]

Recently two types of linear recurrence relations satisfied by the \( c_m \) have been found. The first [5] is an inductive formula for \( c_m \) depending on all the preceding \( c_n \); the second [4] gives \( c_m \) in terms of the preceding \( q \) coefficients. An alternative method of obtaining a recursion formula of the latter type has been pointed out by E. M. Wright [6].

J. G. van der Corput [8] has shown that the inverse of \( g(w) \) possesses the asymptotic development

\[
(g(w))^{-1} = ((2\pi)^{1/2})^{\alpha - 1} \alpha^{-aw + \beta + 1/2} \left[ \sum_{m=0}^{M} \gamma_m \Gamma(\alpha w + \beta - m) \right.
\]

\[
+ O(\Gamma(\alpha w + \beta - M - 1)) \Big] ,
\]

in the sector \( |\arg w| < \pi - \epsilon \). Moreover, van der Corput has extended

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the method of [5] to obtain a linear recursion formula for \( \gamma_m, \gamma_0 = 1 \), depending on all the preceding \( \gamma_n \). The purpose here is to present a general method which when applied to \( g(w) \) and \( (g(w))^{-1} \) yields a finite recursion formula of fixed length \( q \) for the \( c_m \) and \( \gamma_m \) respectively.\(^1\)

2. General factorial expansion. Let us consider the function

\[
F(x) = \prod_{j=0}^{q} \frac{\Gamma(x + s_j)}{\Gamma(x + r_j)},
\]

where the \( s_j \) and \( r_j \) are independent of \( x \). From Stirling’s formula it is easy to show that there exist constants \( A_m, A_0 = 1 \), such that in the sector \(|\arg x| \leq \pi - \epsilon F(x) = \sum_{m=0}^{M} A_m x^{-b-m} + O(x^{-b-M-1})\), where

\[
b = \sum_{j=0}^{q} (r_j - s_j).
\]

The proof which is similar to the proofs cited for (1.2) is omitted. We immediately have the following result which will be stated in the form of a theorem.

**Theorem 2.1.** Let \( d_m \) and \( a \) be given complex numbers, \(|\arg a| < \pi/2\). Then there exist constants \( B_m, B_0 = 1 \), such that

\[
F(x/a) = a^b \sum_{m=0}^{M} B_m \Gamma(x + d_m - m)/\Gamma(x + d_m + b) + O(x^{-b-M-1})
\]

in the sector \(|\arg x| < \pi/2 + \epsilon \) provided \( \epsilon < \pi/2 - |\arg a| \).

The problem is to determine an efficient method for computing the coefficients \( B_m \). For this purpose we shall need the general result of the next section.

3. A general theorem. Let \( G(x) \) denote a function analytic in some sector \(|\arg (x - x_0)| \leq \pi/2 + \epsilon, (\epsilon < \pi/2)\), such that in this sector

\[
G(x) = O(x^{-h}),
\]

where \( h \) denotes a constant. For such a function, we define

\[
\psi(t) = L\{G\} = (2\pi i)^{-1} \int_{C} \lambda^t G(x) dx, \quad (1 \leq t < \infty \text{ if } \lambda > 0),
\]

\[
(0 < t \leq 1 \text{ if } \lambda < 0),
\]

where \( \lambda (\neq 0) \) is real and \( C \) denotes the path \(|\arg (x - x_0)| = \pi/2 + \epsilon\) orientated in the upward direction; clearly, \( \psi(t) \) does not depend on

\(^1\) We had originally demonstrated the result for \( \gamma_m \). This method of giving the general result was suggested by the referee.
the choice of $\epsilon$ or $x_0$. Further, $\psi(t)$ is infinitely differentiable, in fact

$$
(3.3) \quad \psi^{(k)}(t) = \left(\frac{d}{dt}\right)^k L\{G\} = t^{-k}L\{\lambda x(\lambda x - 1) \cdots (\lambda x - k + 1)G(x)\}.
$$

Finally, we note the following result:

**Theorem 3.1.** $G(x) = O(x^{-h})$ implies that near $t=1$

$$
(3.4) \quad \psi^{(k)}(t) = O(\left| t - 1 \right|^{h-k-1}),
$$

provided $k < \text{Re} (h) - 1$.

**Proof.** In the integral representation of $\psi^{(k)}(t)$ replace the curve $C$ by $\text{Re} (x) = u = u_0 = \text{Re} (x_0)$ and set $x = u + iv$ to obtain

$$
\psi^{(k)}(t) = (2\pi)^{-1}t^{h-k} \int_{-\infty}^{+\infty} t^{\lambda x}(\lambda u + i\lambda v)\lambda u - 1 + i\lambda v \cdots 
$$

$$
(\lambda u - k + 1 + i\lambda v)G(u + iv)dv.
$$

There exists a constant $K$, independent of $u$ and $v$, such that

$$
\left| \psi^{(k)}(t) \right| < K \int_{-\infty}^{+\infty} t^{h-k}((u^2 + v^2)^{1/2})^{\text{Re} h} dv.
$$

Substitute $v^2 = wu^2$ to write the last relation in the form

$$
\left| \psi^{(k)}(t) \right| < \left( K/2 \right) u^{k-\text{Re} h+1} \int_{-\infty}^{+\infty} t^{h-k}((1 + w)^{1/2})^{\text{Re} h} w^{-1/2} dw.
$$

Since the integral is convergent, the assertion follows immediately upon setting $u = \left| t - 1 \right|^{-1}$.

**4. The function $\phi(t)$.** First apply Theorem 3.1 to the order term in (2.3) to find that, near $t=1$,

$$
\left(\frac{d}{dt}\right)^h L\left\{F(x/a) - a^b \sum_{m=0}^{M} B_m \Gamma(x + d_m - m)/\Gamma(x + d_m + b)\right\}
$$

$$
= O(\left| t - 1 \right|^{b+M-h}),
$$

provided $k < M + \text{Re} (b)$. It now follows upon introducing

$$
(4.1) \quad \phi(t) = L\{F(x/a)\}, \quad (1 \leq t < \infty \text{ if } \lambda > 0),
$$

$$
(0 < t \leq 1 \text{ if } \lambda < 0),
$$

and using the well-known relation [1, p. 261]

$$
L\left\{\Gamma(x + d_m - m)/\Gamma(x + d_m + b)\right\}
$$

$$
= (\Gamma(m + b))^{-1}t^{-\lambda (d_m + b - 1)}(\lambda - 1)^{m+b-1},
$$
that
\[
\phi^{(k)}(t) = a_b \sum_{m=0}^{M} B_m(\Gamma(m + b))^{-1} \left( \frac{d}{dt} \right)^{k} [t^{-\lambda}(d_{m+b-1})(t^\lambda - 1)^{m+b-1}] + O(\left| t - 1 \right|^{b+M-k}),
\]
provided \( k < M + \text{Re} (b) \).

Now note that by applying the recurrence relation \( \Gamma(x+1) = x\Gamma(x) \) to (2.1), we obtain
\[
(4.3) \quad F(x/a + 1) \prod_{j=0}^{q} (x + ar_j) = F(x/a) \prod_{j=0}^{q} (x + as_j).
\]

There exist constants \( C_k \) and \( E_k \) \( (k = 0, 1, \ldots, q+1) \) such that
\[
(4.4) \quad \prod_{j=0}^{q} (x + as_j) = \sum_{k=0}^{q+1} C_k \lambda x(\lambda x - 1) \cdots (\lambda x - k + 1),
\]
\[
\prod_{j=0}^{q} (x + ar_j) = \sum_{k=0}^{q+1} E_k (\lambda x + \lambda a)(\lambda x + \lambda a - 1) \cdots (\lambda x + \lambda a - k + 1).
\]

Relations (4.3) and (4.4) are used to prove the assertion which follows.

**Theorem 4.1.** The function \( \phi(t) \) defined by (4.1) satisfies the linear differential equation
\[
(4.5) \quad \sum_{k=0}^{q+1} (t^{k+\lambda a}C_k - t^kE_k)\phi^{(k)}(t) = 0,
\]
where
\[
C_k = (k!)^{-1} \prod_{j=0}^{q} (as_j + k/\lambda) - \sum_{n=0}^{k-1} C_n/(k - n)!,
\]
\[
E_k = (k!)^{-1} \prod_{j=0}^{q} (ar_j - a + k/\lambda) - \sum_{n=0}^{k-1} E_n/(k - n)!.
\]

In particular
\[
C_{q+1} = \lambda^{-q-1}; \quad C_q = \lambda^{-q} \left[ a \sum_{j=0}^{q} s_j + q(q + 1)(2\lambda)^{-1} \right],
\]
\[
E_{q+1} = \lambda^{-q-1}; \quad E_q = \lambda^{-q} \left[ a \sum_{j=0}^{q} r_j + (q - 2a\lambda)(q + 1)(2\lambda)^{-1} \right].
\]

**Proof.** From (3.3) and (4.1) we may write
\[ \sum_{k=0}^{q+1} E_k t^k \phi^{(k)}(t) = L \left\{ F(x/a) \sum_{k=0}^{q+1} E_k \lambda x (\lambda x - 1) \cdots (\lambda x - k + 1) \right\} \]
\[ = t^{\lambda a} L \left\{ F(x/a + 1) \sum_{k=0}^{q+1} E_k (\lambda x + \lambda a) (\lambda x + \lambda a - 1) \cdots (\lambda x + \lambda a - k + 1) \right\} \]
\[ = t^{\lambda a} \sum_{k=0}^{q+1} C_k t^k \phi^{(k)}(t), \]
which proves assertion (4.5). Equations (4.6) are obtained immediately upon substituting the appropriate values of \( x \) into the identities (4.4). Relations (4.7) are obtained by comparing the coefficients of \( x^{q+1} \) and \( x^q \) in the identities (4.4).

5. Recursion formulae. A general technique for obtaining the constants \( b_m \) in (2.3) is to substitute the asymptotic expansions (4.2) in (4.5), expand all the resulting terms about \( t=1 \), and equate the coefficients of like powers. This procedure will first be applied to find a finite recursion formula for the constants \( \gamma_m \) which occur in (1.4).

The method requires that we write (1.4) in the form of (2.3) with \( b > q + 1 \). We choose \( b = q+3 \) to be consistent with the notation of [4]. In (1.4) set \( x = \alpha w + \beta + q + 2 \) and use the identity
\[ ((2\pi)^{1/2})^{a-1} \Gamma(x + 1) = \alpha^{q+1/2} \prod_{\mu=1}^{\alpha} \Gamma(x/2 + \mu/2) \]
to find
\[ F_1(x/\alpha) = \alpha^{q+3} \sum_{m=0}^{M} \gamma_m \Gamma(x - q - 2 - m)/\Gamma(x + 1) + O(x^{-q-4-M}), \]
where
\[ F_1(x) = \prod_{j=0}^{q} \Gamma(x + \rho_j - \alpha^{-1}(\beta + q + 2)) \]
\[ \cdot \left( \prod_{i=1}^{p} \Gamma(x + \sigma_i - \alpha^{-1}(\beta + q + 2)) \prod_{\mu=1}^{\alpha} \Gamma(x + \alpha^{-1}\mu) \right)^{-1}. \]
Equation (5.1) is in the form of (2.3) with
(5.2) \[ a = \alpha; \quad b = q + 3; \quad d_m = -q - 2(m = 0, 1, \ldots), \]

\[ s_j = \rho_j - \alpha^{-1} (\beta + q + 2) \quad (j = 0, 1, \ldots, q), \]

(5.3) \[ r_j = \sigma_{j+1} - \alpha^{-1} (\beta + q + 2) \quad (j = 0, 1, \ldots, p - 1), \]

\[ = \alpha^{-1} (j - p + 1) \quad (j = p, \ldots, q). \]

If now we choose \( \lambda = 1 \) in (4.2) we have simply

(5.4) \[ \phi^{(k)}(t) \sim \alpha^{q+3} \sum_{m=0}^{\infty} \frac{\gamma_m(t - 1)^m + q + 2 - k}{\Gamma(m + q + 3 - k)} \quad (1 \leq t < \infty). \]

Substitution of (5.4) into (4.5) yields the desired recurrence relation for the constants \( \gamma_m \).

**Theorem 5.1.** The coefficients \( \gamma_m \) \((m = 0, 1, \ldots)\), occurring in the expansion (1.4), satisfy the following recursion formula of length \( q \):

\[
\gamma_m = (\alpha m)^{-1} (m + 2) ! \cdot \left\{ \sum_{i=1}^{q} \sum_{k=q-i}^{q+1} C_k \left( \begin{array}{c} k \\ q - i \\ \end{array} \right) \frac{\gamma_{m-i}}{(m + q - i + 2 - k)!} \right. \\
- \sum_{i=1}^{p-1} \sum_{k=q-i}^{q+1} E_k \left( \begin{array}{c} k - \alpha \\ q - i - \alpha \\ \end{array} \right) \frac{\gamma_{m-i}}{(m + q - i + 2 - k)!} \right\},
\]

\( (\gamma_0 = 1 \text{ and } \gamma_{-1} = \gamma_{-2} = \cdots = 0) \), where

\[ C_k = (k!)^{-1} \prod_{j=0}^{q} (\alpha \rho_j - \beta - q - 2 + k) - \sum_{n=0}^{k-1} C_n / (k - n) !, \]

(5.6) \[ E_k = (k!)^{-1} \prod_{j=0}^{p-1} (\alpha \sigma_{j+1} - \beta - q - 2 - \alpha + k) \prod_{j=p}^{q} (j - q + k) \]

\[ - \sum_{n=0}^{k-1} E_n / (k - n) !. \]

In particular,

\[ C_{q+1} = 1; \quad C_q = \alpha \sum_{j=0}^{q} \rho_j - (q + 1)(\beta + q + 2) + q(q + 1)/2, \]

(5.7) \[ E_{q+1} = 1; \quad E_q = \alpha \sum_{i=1}^{p} \sigma_i - p(\alpha + \beta + q + 2) + q(q + 1)/2 - \alpha(\alpha - 1)/2. \]

**Proof.** Equations (5.6) and (5.7) are immediately obtained by the substitution of (5.2) and (5.3) into (4.6) and (4.7). To find the recursion formula (5.5) set \( \lambda = 1 \) and employ the binomial theorem to write (4.5) in the form
By (5.4) the coefficient of \( t^{m+2} \) in the asymptotic expansion for 
\( (t-1)^{k-i\phi(k)}(t) \) is

\[
\alpha^{q+3}\gamma_{m-q+j}((m+j+2-k)!)^{-1},
\]

so that, for each integer \( m \),

\[
\sum_{j=0}^{q+1} \sum_{k=j}^{q+1} C_k \left( \begin{array}{c} k \\ j \end{array} \right) \gamma_{m-q+j}((m+j+2-k)!)^{-1} - \sum_{j=\alpha}^{q+1} \sum_{k=j}^{q+1} E_k \left( \begin{array}{c} k - \alpha \\ j - \alpha \end{array} \right) \gamma_{m-q+j}((m+j+2-k)!)^{-1} = 0,
\]

provided we agree that \( \gamma_m = 0 \) for \( m < 0 \). From (5.7), the coefficients of \( \gamma_{m+1} \) and \( \gamma_m \) are 0 and \( \alpha m / (m+2)! \) respectively. The above formula therefore reduces to (5.5) upon solving for \( \gamma_m \) and setting \( i = q - j \).

The general method may also be applied to obtain a recursion formula for the constants \( c_m \) which appear in (1.2). In this case set 
\( x = \alpha w + \beta - q - 3 \) and write (1.2) as

\[
F_2(x/\alpha) = \alpha^{q+3} \sum_{m=0}^{M} c_m \Gamma(x)/\Gamma(x + q + 3 + m) + O(x^{-q-M}),
\]

where

\[
F_2(x) = \prod_{i=1}^{p} \Gamma(x + \sigma_i - \alpha^{-1}(\beta - q - 3)) \\
\cdot \prod_{\mu=1}^{\alpha} \Gamma(x + \alpha^{-1}(\mu - 1)) / \prod_{j=0}^{\rho} \Gamma(x + \rho_j - \alpha^{-1}(\beta - q - 3)).
\]

Now (5.8) is in the form of (2.3) with

\[
(5.9) \quad a = \alpha; \quad b = q + 3; \quad d_m = m,
\]

\[
(5.10) \quad s_j = \sigma_{j+1} - \alpha^{-1}(\beta - q - 3) \quad (j = 0, 1, \cdots, p - 1)
\]

\[
(5.10) \quad r_j = \rho_j - \alpha^{-1}(\beta - q - 3) \quad (j = 0, 1, \cdots, q).
\]

If we choose \( \lambda = -1 \), then (4.2) and (4.5) become simply

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\[ \phi^{(k)}(t) \sim \alpha^{q+3} \sum_{m=0}^{\infty} (-1)^{m+q} C_m (t - 1)^{m+q+2-k}/\Gamma(m + q + 3 - k) \]

\[ (0 < t \leq 1) \]

and

\[ \sum_{k=0}^{q+1} (t^{k-a} C_k - t^{k} E_k) \phi^{(k)}(t) = 0 \]

respectively. The recursion formula for the \( c_m \) obtained by substituting (5.11) into (5.12) is precisely the result given in [4].

6. Remark. When Theorem (5.1) is applied to the special case of

\[ (g(w))^{-1} = \Gamma(w + \rho_0) \Gamma(w + \rho_1) \]

the result is

\[ \gamma_m (8m)^{-1} [4(\rho_1 - \rho_0)^2 - (2m - 1)^2] \gamma_{m-1}. \]

On the other hand, van der Corput's formula [8, Equation 8] gives

\[ \gamma_1 = (8)^{-1} [4(\rho_1 - \rho_0)^2 + 71] \gamma_0 \]

for the second coefficient. This discrepancy has been resolved by H. O. Pollak who has computed \( \gamma_1 \) for this special case by substituting Stirling's formula into (6.1) directly. The result is in agreement with (6.2).

Bibliography
