COEFFICIENTS IN CERTAIN ASYMPTOTIC FACTORIAL EXPANSIONS

T. D. RINEY

1. Introduction. Let $p$ and $q$ denote integers with $0 \leq p \leq q$, then it is known from [2; 3] and [7] that the function

$$g(w) = \prod_{i=1}^{p} \frac{\Gamma(w + \sigma_i)}{\prod_{j=0}^{q} \Gamma(w + \rho_j)},$$

where the $\sigma_i$ and $\rho_j$ are independent of $w$, admits for large $|w|$ in the sector $|\arg w| < \pi - \epsilon$, an asymptotic expansion of the form

$$g(w) = ((2\pi)^{1/2})^{1-\sigma} \alpha^{\omega+\beta-1/2} \sum_{m=0}^{M} \frac{c_m}{\Gamma(\alpha w + \beta + m)}$$

$$+ O(1/\Gamma(\alpha w + \beta + M + 1)).$$

Here, and in the sequel, $M$ denotes a non-negative integer independent of $w$, $\epsilon$ any positive constant and

$$c_0 = 1; \quad \alpha = q + 1 - p; \quad \beta = \sum_{j=0}^{q} \rho_j - \sum_{i=1}^{p} \sigma_i + (1 - \alpha)/2.$$

Recently two types of linear recurrence relations satisfied by the $c_m$ have been found. The first [5] is an inductive formula for $c_m$ depending on all the preceding $c_n$; the second [4] gives $c_m$ in terms of the preceding $q$ coefficients. An alternative method of obtaining a recursion formula of the latter type has been pointed out by E. M. Wright [6].

J. G. van der Corput [8] has shown that the inverse of $g(w)$ possesses the asymptotic development

$$(g(w))^{-1} = ((2\pi)^{1/2})^{\omega-\alpha} \alpha^{-\omega-\beta+1/2} \sum_{m=0}^{M} \gamma_m \Gamma(\alpha w + \beta - m)$$

$$+ O(\Gamma(\alpha w + \beta - M - 1)),$$

in the sector $|\arg w| < \pi - \epsilon$. Moreover, van der Corput has extended

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the method of [5] to obtain a linear recursion formula for \( \gamma_m, \gamma_0 = 1 \), depending on all the preceding \( \gamma_n \). The purpose here is to present a general method which when applied to \( g(w) \) and \( (g(w))^{-1} \) yields a finite recursion formula of fixed length \( q \) for the \( c_m \) and \( \gamma_m \) respectively.\(^1\)

2. General factorial expansion. Let us consider the function

\[
F(x) = \prod_{j=0}^{q} \frac{\Gamma(x + s_j)}{\Gamma(x + r_j)},
\]

where the \( s_j \) and \( r_j \) are independent of \( x \). From Stirling’s formula it is easy to show that there exist constants \( A_m, A_0 = 1 \), such that in the sector \( |\arg x| \leq \pi - \epsilon \),

\[
\begin{align*}
F(x) &= \sum_{m=0}^{M} A_m x^{-b-m} + O(x^{-b-M-1}), \\
\end{align*}
\]

where

\[
b = \sum_{j=0}^{q} (r_j - s_j).
\]

The proof which is similar to the proofs cited for (1.2) is omitted. We immediately have the following result which will be stated in the form of a theorem.

**Theorem 2.1.** Let \( d_m \) and \( a \) be given complex numbers, \( |\arg a| < \pi/2 \). Then there exist constants \( B_m, B_0 = 1 \), such that

\[
F(x/a) = a^b \sum_{m=0}^{M} B_m \frac{\Gamma(x + d_m - m)}{\Gamma(x + d_m + b)} + O(x^{-b-M-1})
\]

in the sector \( |\arg x| < \pi/2 + \epsilon \) provided \( \epsilon < \pi/2 - |\arg a| \).

The problem is to determine an efficient method for computing the coefficients \( B_m \). For this purpose we shall need the general result of the next section.

3. A general theorem. Let \( G(x) \) denote a function analytic in some sector \( |\arg (x - x_0)| \leq \pi/2 + \epsilon \), \( \epsilon < \pi/2 \), such that in this sector

\[
G(x) = O(x^{-h}),
\]

where \( h \) denotes a constant. For such a function, we define

\[
\psi(t) = L\{G\} = (2\pi i)^{-1} \int_C \lambda^t G(x) dx,
\]

where \( \lambda \neq 0 \) is real and \( C \) denotes the path \( |\arg (x - x_0)| = \pi/2 + \epsilon \) orientated in the upward direction; clearly, \( \psi(t) \) does not depend on

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\(^1\) We had originally demonstrated the result for \( \gamma_m \). This method of giving the general result was suggested by the referee.
the choice of $\epsilon$ or $x_0$. Further, $\psi(t)$ is infinitely differentiable, in fact

\begin{equation}
\psi^{(k)}(t) = \left( \frac{d}{dt} \right)^k L \{ G \} = t^{-k} L \{ \lambda x(\lambda x - 1) \cdots (\lambda x - k + 1) G(x) \}.
\end{equation}

Finally, we note the following result:

**Theorem 3.1.** $G(x) = O(x^{-h})$ implies that near $t = 1$

\begin{equation}
\psi^{(k)}(t) = O(\left| t - 1 \right|^{h-k-1}),
\end{equation}

provided $k < \text{Re} (h) - 1$.

**Proof.** In the integral representation of $\psi^{(k)}(t)$ replace the curve $C$ by $\text{Re}(x) = u_0 = \text{Re}(x_0)$ and set $x = u + iv$ to obtain

\begin{equation}
\psi^{(k)}(t) = (2\pi)^{-1/2} t^{\lambda u - k} \int_{-\infty}^{+\infty} t^{i\lambda v}(\lambda u + i\lambda v)(\lambda u - 1 + i\lambda v) \cdots \\
(\lambda u - k + 1 + i\lambda v)G(u + iv)dv.
\end{equation}

There exists a constant $K$, independent of $u$ and $v$, such that

\[ |\psi^{(k)}(t)| < K \int_{-\infty}^{+\infty} t^{\lambda u - k}((u^2 + v^2)^{1/2})^{h - \text{Re} h} dv. \]

Substitute $v^2 = uw^2$ to write the last relation in the form

\[ |\psi^{(k)}(t)| < (K/2) u^{k - \text{Re} h + 1} \int_{-\infty}^{+\infty} t^{\lambda u - k}((1 + w)^{1/2})^{h - \text{Re} h} w^{-1/2} dw. \]

Since the integral is convergent, the assertion follows immediately upon setting $u = \left| t - 1 \right|^{-1}$.

**4. The function $\phi(t)$.** First apply Theorem 3.1 to the order term in (2.3) to find that, near $t = 1$,

\begin{equation}
\left( \frac{d}{dt} \right)^k L \left\{ F(x/a) - a^b \sum_{m=0}^{M} B_m \Gamma(x + d_m - m)/\Gamma(x + d_m + b) \right\} = O(\left| t - 1 \right|^{b+M-k}),
\end{equation}

provided $k < M + \text{Re} (b)$. It now follows upon introducing

\begin{equation}
\phi(t) = L \{ F(x/a) \}, \quad (1 \leq t < \infty \text{ if } \lambda > 0),
\end{equation}

\[ (0 < t \leq 1 \text{ if } \lambda < 0), \]

and using the well-known relation [1, p. 261]

\begin{equation}
L \{ \Gamma(x + d_m - m)/\Gamma(x + d_m + b) \}
= (\Gamma(m + b))^{-1} t^{-\lambda(d_m+b-1)}(\lambda - 1)^{m+b-1},
\end{equation}

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that
\begin{equation}
\phi^{(k)}(t) = a^b \sum_{m=0}^{M} B_m(\Gamma(m + b))^{-1} \left( \frac{d}{dt} \right)^k \left[ t^{-\lambda(d_m+b-1)}(t^\lambda - 1)^{m+b-1} \right] + O\left( |t - 1|^{b+M-k} \right),
\end{equation}
provided \( k < M + \text{Re}(b) \).

Now note that by applying the recurrence relation \( \Gamma(x + 1) = x\Gamma(x) \) to (2.1), we obtain
\begin{equation}
F(x/a + 1) \prod_{j=0}^{q} (x + ar_j) = F(x/a) \prod_{j=0}^{q} (x + as_j).
\end{equation}

There exist constants \( C_k \) and \( E_k \) \((k = 0, 1, \ldots, q+1)\) such that
\begin{equation}
\prod_{j=0}^{q} (x + as_j) = \sum_{k=0}^{q+1} C_k x(x - 1) \cdots (x - k + 1),
\end{equation}
\begin{equation}
\prod_{j=0}^{q} (x + ar_j) = \sum_{k=0}^{q+1} E_k (x + \lambda a)(x + \lambda a - 1) \cdots (x + \lambda a - k + 1).
\end{equation}

Relations (4.3) and (4.4) are used to prove the assertion which follows.

**Theorem 4.1.** The function \( \phi(t) \) defined by (4.1) satisfies the linear differential equation
\begin{equation}
\sum_{k=0}^{q+1} (t^{k+\lambda a}C_k - t^kE_k)\phi^{(k)}(t) = 0,
\end{equation}
where
\begin{equation}
C_k = (k!)^{-1} \prod_{j=0}^{q} (as_j + k/\lambda) - \sum_{n=0}^{k-1} C_n/(k - n)!,
\end{equation}
\begin{equation}
E_k = (k!)^{-1} \prod_{j=0}^{q} (ar_j - a + k/\lambda) - \sum_{n=0}^{k-1} E_n/(k - n)!.
\end{equation}

In particular
\begin{equation}
C_{q+1} = \lambda^{-q-1}; \quad C_q = \lambda^{-q} \left[ a \sum_{j=0}^{q} s_j + q(q + 1)(2\lambda)^{-1} \right],
\end{equation}
\begin{equation}
E_{q+1} = \lambda^{-q-1}; \quad E_q = \lambda^{-q} \left[ a \sum_{j=0}^{q} r_j + (q - 2a\lambda)(q + 1)(2\lambda)^{-1} \right].
\end{equation}

**Proof.** From (3.3) and (4.1) we may write
\[ \sum_{k=0}^{q+1} E_k t^k \phi^{(k)}(t) = L \left\{ F(x/a) \sum_{k=0}^{q+1} E_k x(x - 1) \cdots (x - k + 1) \right\} \]

\[ = t^{q+1} L \left\{ F(x/a + 1) \sum_{k=0}^{q+1} C_k x(x + 1)(x + 2) \cdots (x + k - 1) \right\} \]

which proves assertion (4.5). Equations (4.6) are obtained immediately upon substituting the appropriate values of \( x \) into the identities (4.4). Relations (4.7) are obtained by comparing the coefficients of \( x^{q+1} \) and \( x^q \) in the identities (4.4).

5. Recursion formulae. A general technique for obtaining the constants \( B_m \) in (2.3) is to substitute the asymptotic expansions (4.2) in (4.5), expand all the resulting terms about \( t = 1 \), and equate the coefficients of like powers. This procedure will first be applied to find a finite recursion formula for the constants \( \gamma_m \) which occur in (1.4).

The method requires that we write (1.4) in the form of (2.3) with \( b > q + 1 \). We choose \( b = q + 3 \) to be consistent with the notation of [4]. In (1.4) set \( x = \alpha w + \beta + q + 2 \) and use the identity

\[ ((2\pi)^{1/2}) \alpha^{x-1} \Gamma(x + 1) = \alpha^{x+1/2} \prod_{\mu=1}^{\alpha} \Gamma(x/2 + \mu/2) \]

to find

\[ (5.1) \quad F_1(x/\alpha) = \alpha^{q+3} \sum_{m=0}^{M} \gamma_m \Gamma(x - q - 2 - m) / \Gamma(x + 1) + O(x^{-q-4-M}), \]

where

\[ F_1(x) = \prod_{j=0}^{q} \Gamma(x + \rho_j - \alpha^{-1}(\beta + q + 2)) \]

\[ \cdot \left( \prod_{i=1}^{p} \Gamma(x + \sigma_i - \alpha^{-1}(\beta + q + 2)) \prod_{\mu=1}^{\alpha} \Gamma(x + \alpha^{-1}\mu) \right)^{-1}. \]

Equation (5.1) is in the form of (2.3) with
\(a = \alpha; \quad b = q + 3; \quad d_m = -q - 2(m = 0, 1, \ldots),\)
\[s_j = \rho_j - \alpha^{-1}(\beta + q + 2) \quad (j = 0, 1, \ldots, q),\]
\[r_j = \sigma_{j+1} - \alpha^{-1}(\beta + q + 2) \quad (j = 0, 1, \ldots, p - 1),\]
\[= \alpha^{-1}(j - p + 1) \quad (j = p, \ldots, q).\]

If now we choose \(\lambda = 1\) in (4.2) we have simply
\[\phi^{(k)}(t) \sim \alpha^{q+3} \sum_{m=0}^{\infty} \frac{\gamma_m(t-1)^{m+q+2-k}}{\Gamma(m + q + 3 - k)} \quad (1 \leq t < \infty).\]

Substitution of (5.4) into (4.5) yields the desired recurrence relation
for the constants \(\gamma_m\).

**Theorem 5.1.** The coefficients \(\gamma_m (m = 0, 1, \ldots)\), occurring in the
expansion (1.4), satisfy the following recursion formula of length \(q:\)
\[\gamma_m = (\alpha m)^{-1}(m + 2)! \cdot \left\{ \sum_{i=1}^{q} \sum_{k=q-i}^{q+1} C_k \binom{k}{q-i} \gamma_{m-i}/(m + q - i + 2 - k)! \right\},\]
\[\left\{ \sum_{i=1}^{p-1} \sum_{k=q-i}^{q+1} E_k \binom{k-\alpha}{q-i-\alpha} \gamma_{m-i}/(m + q - i + 2 - k)! \right\},\]
\((\gamma_0 = 1 \text{ and } \gamma_{-1} = \gamma_{-2} = \ldots = 0), \text{ where}\)
\[C_k = (k!)^{-1} \prod_{j=0}^{q} (\alpha \rho_j - \beta - q - 2 + k) - \sum_{n=0}^{k-1} C_n/(k - n)!,\]
\[E_k = (k!)^{-1} \prod_{j=0}^{p-1} (\alpha \sigma_{j+1} - \beta - q - 2 - \alpha + k) \prod_{j=p}^{q} (j - q + k) \]
\[- \sum_{n=0}^{k-1} E_n/(k - n)!.\]

*In particular,*
\[C_{q+1} = 1; \quad C_q = \alpha \sum_{j=0}^{q} \rho_j - (q + 1)(\beta + q + 2) + q(q + 1)/2,\]
\[E_{q+1} = 1; \quad E_q = \alpha \sum_{i=1}^{p} \sigma_i - p(\alpha + \beta + q + 2) + q(q + 1)/2 - \alpha(\alpha - 1)/2.\]

**Proof.** Equations (5.6) and (5.7) are immediately obtained by the
substitution of (5.2) and (5.3) into (4.6) and (4.7). To find the recur-
sion formula (5.5) set \(\lambda = 1\) and employ the binomial theorem to write
(4.5) in the form
\[
\sum_{j=0}^{q+1} \sum_{k=j}^{q+1} C_k \left( \begin{array}{c} k \\ j \end{array} \right) (t - 1)^{k-\ell} \phi^{(k)}(t) - \sum_{j=0}^{q+1} \sum_{k=j}^{q+1} E_k \left( \begin{array}{c} k - \alpha \\ j - \alpha \end{array} \right) (t - 1)^{k-\ell} \phi^{(k)}(t) = 0.
\]

By (5.4) the coefficient of \( t^{m+2} \) in the asymptotic expansion for \((t - 1)^{k-\ell} \phi^{(k)}(t)\) is

\[
\alpha^{q+3} \gamma_{m-q+j}((m + j + 2 - k)!)^{-1},
\]

so that, for each integer \( m \),

\[
\sum_{j=0}^{q+1} \sum_{k=j}^{q+1} C_k \left( \begin{array}{c} k \\ j \end{array} \right) \gamma_{m-q+j}((m + j + 2 - k)!)^{-1} - \sum_{j=0}^{q+1} \sum_{k=j}^{q+1} E_k \left( \begin{array}{c} k - \alpha \\ j - \alpha \end{array} \right) \gamma_{m-q+j}((m + j + 2 - k)!)^{-1} = 0,
\]

provided we agree that \( \gamma_m = 0 \) for \( m < 0 \). From (5.7), the coefficients of \( \gamma_{m+1} \) and \( \gamma_m \) are 0 and \( \alpha m/(m+2)! \) respectively. The above formula therefore reduces to (5.5) upon solving for \( \gamma_m \) and setting \( i = q - j \).

The general method may also be applied to obtain a recursion formula for the constants \( c_m \) which appear in (1.2). In this case set \( x = \alpha w + \beta - q - 3 \) and write (1.2) as

\[
(5.8) \quad F_2(x/\alpha) = \alpha^{q+3} \sum_{m=0}^{M} c_m \Gamma(x)/\Gamma(x + q + 3 + m) + O(x^{-q-M}),
\]

where

\[
F_2(x) = \prod_{i=1}^{p} \Gamma(x + \sigma_i - \alpha^{-1}(\beta - q - 3)) \cdot \prod_{\mu=1}^{\alpha} \Gamma(x + \alpha^{-1}(\mu - 1)) \begin{array}{c} \\
\prod_{j=0}^{q} \Gamma(x + \rho_j - \alpha^{-1}(\beta - q - 3)).
\end{array}
\]

Now (5.8) is in the form of (2.3) with

\[
\begin{align*}
(5.9) & \quad a = \alpha; \quad b = q + 3; \quad d_m = m, \\
& \quad s_j = \sigma_{j+1} - \alpha^{-1}(\beta - q - 3) \quad (j = 0, 1, \ldots, p - 1) \\
(5.10) & \quad r_j = \rho_j - \alpha^{-1}(\beta - q - 3) \quad (j = 0, 1, \ldots, q). \end{align*}
\]

If we choose \( \lambda = -1 \), then (4.2) and (4.5) become simply
\[ \phi^{(k)}(t) \sim \alpha^{q+3} \sum_{m=0}^{\infty} \frac{(-1)^{m+q} c_m(t-1)^{m+q+2-k}}{\Gamma(m+q+3-k)} \quad (0 < t \leq 1) \]  

(5.11)

and

\[ \sum_{k=0}^{q+1} (t^{k-a} C_k - t^k E_k) \phi^{(k)}(t) = 0 \]

(5.12)

respectively. The recursion formula for the \( c_m \) obtained by substituting (5.11) into (5.12) is precisely the result given in [4].

6. **Remark.** When Theorem (5.1) is applied to the special case of

(6.1) \[ (g(w))^{-1} = \Gamma(w + \rho_0) \Gamma(w + \rho_1) \]

the result is

(6.2) \[ \gamma_m (8m)^{-1} [4(\rho_1 - \rho_0)^2 - (2m - 1)^2] \gamma_{m-1}. \]

On the other hand, van der Corput’s formula [8, Equation 8] gives

\[ \gamma_1 = (8)^{-1} [4(\rho_1 - \rho_0)^2 + 71] \gamma_0 \]

for the second coefficient. This discrepancy has been resolved by H. O. Pollak who has computed \( \gamma_1 \) for this special case by substituting Stirling’s formula into (6.1) directly. The result is in agreement with (6.2).

**Bibliography**
