ON THE NOTION OF ANALYTICITY

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Let $E_n$ be a normed $n$-dimensional vector space over a field $\Gamma$. We denote by $a, b, \cdots$ the elements of $E_n$, with $\alpha, \beta, \cdots$ the elements of $\Gamma$; $a = \alpha_1e_1 + \cdots + \alpha_ne_n \sim \{\alpha_1, \cdots, \alpha_n\}$ where $e_1, \cdots, e_n$ is a basis of $E_n$. The norm in $E_n$ we denote by $\| \ |$.

By $a = L \circ b$ we denote a linear homogeneous transformation on $E_n$ to $E_n$, i.e. $a \in E_n$ is defined for every $b \in E_n$; $L \circ (\lambda b) = \lambda L \circ b$; $L \circ (b_1 + b_2) = L \circ b_1 + L \circ b_2$ for every $\lambda, b, b_1, b_2$. Naturally, given a basis $e_1, \cdots, e_n$ of $E_n$, $L$ can be represented as a square matrix: $L \sim [\lambda_{\mu\nu}]$ such that $\alpha_\mu = \lambda_{\mu\beta_1 + \cdots + \lambda_{\mu\beta_n} (\mu = 1, \cdots, n)}$ for $a = L \circ b, a \sim \{\alpha_1, \cdots, \alpha\}, b = \{\beta_1, \cdots, \beta_n\}$.

Let $y = f(x)$ be a function on $E_n$ to $E_n$, i.e. $y \in E_n$ for some $x \in E_n$. $f(x)$ is called differentiable (in the sense of Fréchet) at $x = x_0$ if for $h \in E_n, \| h \|_n \to 0$ there is a $L = L(x_0)$ such that

(1) $f(x_0 + h) = f(x_0) + L(x_0) \circ h + r(h)\; r(h) \in E_n, \| r(h) \|_n = o(\| h \|_n)$.

If, with respect to the basis $e_1, \cdots, e_n$ it is $f \sim \{\phi_1, \cdots, \phi_n\}$, $x \sim \{\xi_1, \cdots, \xi_n\}, L \sim [\lambda_{\mu\nu}]$ then obviously

(2) $\lambda_{\mu\nu} = \partial \phi_\mu / \partial \xi_\nu \; (x = x_0)$.

We call $\otimes$ a product on $E_n$ if to every $a \in E_n, b \in E_n$ there exists $c = a \otimes b \in E_n$ such that $a \otimes b = L_a \circ b = L_b \circ a$. For some fixed basis $e_1, \cdots, e_n$ with $a \sim \{\alpha_1, \cdots, \alpha_n\}, b \sim \{\beta_1, \cdots, \beta_n\}, c \sim \{\gamma_1, \cdots, \gamma_n\}$ such a product can always be realized by $n^3$ constants $\pi_{\rho\mu\nu}$ such that

(3) $\gamma_\rho = \sum_{\mu, \nu = 1}^{n} \pi_{\rho\mu\nu} \alpha_\mu \beta_\nu \; (\rho = 1, \cdots, n)$.

Naturally, by introducing a product in a vector space this vector space becomes an algebra.

If $f(x)$ is a function on $E_n$ to $E_n$ we call it analytic with respect to the product $\otimes$ at $x = x_0$ if it is there differentiable and if there is an $l = l(x_0) \in E_n$ such that

(4) $L(x_0) \circ h = l(x_0) \otimes h$.

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where \( L(x_0) \) is defined by (1). Without referring to the definition of differentiability this means: there is an \( l(x_0) \) such that

\[
f(x_0 + h) = f(x_0) + l(x_0) \otimes h + r(h), \quad r(h) \in E_n,
\]

\[
\left| r(h) \right|_n = o\left( \left| h \right|_n \right) \quad \text{for} \quad \left| h \right|_n \to 0.
\]

It is obvious that this definition of analyticity contains the one for functions of a complex variable.

Let \( l(x_0) \sim \{\lambda_1(x_0), \ldots, \lambda_n(x_0)\} \), \( h \sim \{\eta_1, \ldots, \eta_n\} \) for a basis \( e_1, \ldots, e_n \). Putting \( l(x_0) \otimes h = k \), \( k \sim \{\kappa_1, \ldots, \kappa_n\} \) we have \( \kappa_\rho = \sum_{\mu=1}^n \pi_{\rho\mu} \lambda_\mu(x_0) \eta_\nu \) (\( \rho = 1, \ldots, n \)). On the other hand, the \( \rho \)th component of \( L(x_0) \circ h \) is given by

\[
\lambda_\rho \eta_1 + \cdots + \lambda_n \eta_n = \partial \phi_\rho / \partial \xi_1 \cdot \eta_1 + \cdots + \partial \phi_\rho / \partial \xi_n \cdot \eta_n.
\]

Instead of (4) we have therefore

\[
\partial \phi_\rho / \partial \xi_\nu = \pi_{\rho1} \lambda_1(x_0) + \cdots + \pi_{\rho n} \lambda_n(x_0) \quad (\rho, \nu = 1, \ldots, n).
\]

We put

\[
\Pi_\nu = \text{Det} \pi_{\rho\nu} = \begin{vmatrix} \pi_{11\nu} & \cdots & \pi_{1\nu} \\ \vdots & \ddots & \vdots \\ \pi_{n1\nu} & \cdots & \pi_{nn\nu} \end{vmatrix} \quad (\nu = 1, \ldots, n)
\]

and denote by \( \Pi_{\nu \rho} \) the determinant we get from \( \Pi_\nu \) by replacing its \( \rho \)th column by \( \partial \phi_1 / \partial \xi_\nu, \ldots, \partial \phi_n / \partial \xi_\nu \). It follows

\[
\Pi_{1\rho} / \Pi_1 = \cdots = \Pi_{n\rho} / \Pi_n (= \lambda_\rho) \quad (\rho = 1, \ldots, n),
\]

provided \( \Pi_\nu \neq 0 \) (\( \nu = 1, \ldots, n \)). If some of these determinants vanish, \( \Pi_1 = \cdots = \Pi_k = 0, k \leq n \) say, then (7) has to be replaced by

\[
\Pi_{1\rho} = \cdots = \Pi_{k\rho} = 0, \quad \Pi_{k+1,\rho} / \Pi_{k+1} = \cdots = \Pi_{n\rho} / \Pi_n \quad (\rho = 1, \ldots, n).
\]

The equations (7) viz (8) form a necessary condition for the analyticity of \( f(x) \) at \( x = x_0 \). We are therefore entitled to consider them as a generalization of the Cauchy-Riemann differential equations.

In the case \( n = 2 \) and complex multiplication, the equations (7) yield the Cauchy-Riemann equations. In the case \( n = 4 \), \( E_n \) being a real space and the product being defined according to the multiplication of quaternions, these equations read
\[ \phi_{11} = \phi_{22} = \phi_{33} = \phi_{44}, \]
\[ \phi_{21} = -\phi_{12} = \phi_{43} = -\phi_{34}, \]
\[ \phi_{31} = -\phi_{42} = -\phi_{13} = \phi_{24}, \]
\[ \phi_{41} = \phi_{32} = -\phi_{23} = -\phi_{14}. \]

where we have abbreviated \( \partial \phi_{\nu} / \partial \xi_{\mu} \) by \( \phi_{\nu\mu} \). In the case \( n = 3 \), \( E_n \) being real and the product being the ordinary vector product, the function \( f(\xi_1, \xi_2, \xi_3) = \{ \xi_2 - \xi_3, \xi_3 - \xi_1, \xi_1 - \xi_2 \} \) turns out to be everywhere analytic.

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