DIFEOOMORPHISMS OF THE 2-SPHERE
STEPHEN SMALE

The object of this paper is to prove the theorem.

THEOREM A. The space $\Omega$ of all orientation preserving $C^\infty$ diffeomorphisms of $S^2$ has as a strong deformation retract the rotation group $SO(3)$.

Here $S^2$ is the unit sphere in Euclidean 3-space, the topology on $\Omega$ is the $C^r$ topology $\infty \geq r > 1$ (see [4]) and a diffeomorphism is a differentiable homeomorphism with differentiable inverse.

The method of proof uses

THEOREM B. The space $\mathcal{F}$ ($C^r$ topology) of $C^\infty$ diffeomorphisms of the unit square which are the identity in some neighborhood of the boundary is contractible to a point.

The analogue of Theorem A for the topological case was proved by H. Kneser [2]. The problem in his case seems to be of a different nature from the differentiable case. J. Munkres [3] has proved that $\Omega$ is arcwise connected.

Conversations with R. Palais have been helpful in the preparation of this paper.

Let $I^2$ be the square in the Euclidean plane $E^2$ with coordinate $(t, x)$ such that $(t, x) \in I^2$ if $0 \leq t \leq 1$ and $0 \leq x \leq 1$. Let $\bar{e}: I^2 \to I^2$ denote the identity diffeomorphism and $\mathcal{F}$, the space of diffeomorphisms, with the $C^r$ topology, of $I^2$ onto $I^2$ which agree with $\bar{e}$ on some neighborhood of $\partial I^2$, the boundary of $I^2$. The $C^r$ topology is such that two maps are close with respect to it if they are close and their first $r$ derivatives are close. See R. Thom [4] for details. We assume that $r$ is fixed in this paper, $\infty \geq r > 1$, and that all function spaces considered possess the $C^r$ topology. We further assume that all diffeomorphisms are $C^\infty$.

Let $I_1 \subset I^2$ denote the subset $\{(t, x) | t = 1\}$, $df_p$ be the differential of a diffeomorphism $f$ at $p$, and $u_0$ be the vector $(1, 0)$ in $E^2$ considered as its own tangent vector space. Then denote by $\mathcal{E}$ the space of diffeomorphisms of $I^2$ onto $I^2$ such that if $f \in \mathcal{E}$, then (a) $f = \bar{e}$ on some neighborhood of $I_1$, and (b) $df_p(u_0) = u_0$ for all $p$ in some neighborhood of $I_1$.

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Let $e_0: I^2 \to u_0$ be the constant map and define $S$ to be the space with the compact open topology of maps of $I^2$ into $S$ which agree with $e_0$ in some neighborhood of $I^2$, where $S = E^2 - (0, 0)$.

A map $\phi: E \to S$ is defined as follows:

$$\phi(f)(t, x) = df^{-1}(t, x)(u_0), \quad f \in E.$$

**Lemma 1.** There is a homotopy $\Phi: E \to S$ such that for each $f \in E$,

- (a) $\Phi(f)(t, x)$ is $C^\infty$ in $(v, x, t)$,
- (b) $\Phi(0) = e_0$,
- (c) $\Phi^1 = \Phi$, and
- (d) $\Phi(\varepsilon) = e_0$.

**Proof.** Let $p: R \to S$ be the covering map where $R$ is the universal covering space of $S$, and let $\bar{u} \in p^{-1}(u_0)$. Let $T_v: R \to R$ be a differentiable contraction of $R$ to $\bar{u}$.

Define now a homotopy $h_v: S \to S$ by $h_v(f)(x) = pT_v(f(x))$ where $f$ is the unique lifting of $f$ taking $I^2$ into $\bar{u}$. Then it is easily checked that $\Phi^v = h_v \circ \Phi$ may be taken as our desired homotopy.

**Lemma 2.** There is a homotopy $H_v: S \to S$ such that for each $f \in E$,

- (a) $H_v(f)$ is $C^\infty$ in $(v, x, t)$,
- (b) $H_0(f) = e$,
- (c) $H_1(f) = f$, and
- (d) $H_v(\varepsilon) = e$.

**Proof.** Let $\phi_v(f)(t, x)$ be considered as a vector field on $I^2$ as given by Lemma 1, for each $f \in E$ and $v \in I$, $I$ the unit interval. Let $P_v(f)(u, t_0, x_0)$ be the integral curve of $\phi_v(f)$ with the initial condition $P_v(f)(0, t_0, x_0) = (t_0, x_0)$.

Define $Q_v(f)(t, x) = P_v(f)(t, 0, x)$. Now suppose there were an integral curve $P_v(f)(u, t_0, x_0)$ starting at $(t_0, x_0) = (0, x)$ which did not leave $I^2$. If this were so then it would approach asymptotically some simple closed curve in $I^2$. In the interior of this curve the vector field $\phi_v(f)$ would have to have a singularity. But this is impossible. Thus we conclude that there is a $t$, say $\bar{t}$, with $Q_v(f)(\bar{t}, x)$ meeting $I_1$. This is exactly the part of the proof of Theorem A which does not extend to the case of $S^n$.

Denote the above $\bar{t}$ by $\bar{t}(v, f, x)$. Then $\bar{t}(v, f, x)$ is $C^\infty$ in $(v, x)$ continuous $(v, f, x)$ and positive.

We need the following lemma which can be found for example in [1, p. 172].

**Lemma 3.** Let $g$ be a real lower semi-continuous positive function on a paracompact space $X$. Then there is a real continuous function $h$ on $X$ such that for all $x \in X$, $0 < h(x) < g(x)$.
Let \( g \) be the function on \( \mathbb{S} \) defined by
\[
 g(f) = \min \left\{ \frac{\tilde{l}(v, f, x)}{1 - \tilde{l}(v, f, x)}, 1 \right\} \quad \forall v, x \in I, \tilde{l}(v, f, x) < 1. 
\]

Then let \( \eta \) be the function on \( \mathbb{S} \) given by Lemma 3.

Let \( \gamma \) be a real function on \( \mathbb{S} \times \mathbb{R}, C^\infty \) in \( t \) such that \( \gamma(f, t) = 0 \) for \( t \) in some neighborhood of 0, \( \gamma(f, t) = 1 \) for \( t \) in some neighborhood of 1 and
\[
0 < \frac{d\gamma(f, t)}{dt} < 1 + \eta(f).
\]

We leave to the reader the task of showing that such a function exists.

Now define \( H_\psi: \mathbb{S} \to \mathbb{S} \) by
\[
 H_\psi(f)(t, x) = Q_\psi(f)(t + \gamma(f, t)(\tilde{l}(v, f, x) - 1), x).
\]

We prove now that \( H_\psi(f): I^2 \to I^2 \) is regular (has Jacobian of rank 2). Note that \( H_\psi(f) \) can be written as the composition \( \psi \circ g \) where
\[
g: (t, x) \to (t + \gamma(f, t)(\tilde{l}(v, f, x) - 1), x) = (t', x'),
\]
\[
\psi: (t', x') \to Q_\psi(f)(t', x').
\]

From the choice of \( \eta \) it follows that \( \partial t' / \partial t \neq 0 \), and hence \( g \) is regular.

Now we prove that \( \psi \) is regular. Let
\[
\varphi(u, t, x) = P_\psi(f)(u, t, x) \quad \text{and} \quad \varphi^i(u, t, x), \quad i = 1, 2
\]
be the coordinates of \( \varphi \). Also let \( X(t, x) \) denote the vector field \( \phi_\psi(f) \) with
\[
X(t, x) = X_1(t, x) \frac{\partial}{\partial t} + X_2(t, x) \frac{\partial}{\partial x}.
\]

Then \( \psi(t, x) = \varphi(t, 0, x) \). It is sufficient to prove \( \psi^{-1} \) is differentiable. Let \( \tau(t, x) \) be the unique \( u \) such that \( \varphi^1(u, t, x) = 0 \). Then \( \psi^{-1}(t, x) = (-\tau(t, x), \varphi^1(\tau(t, x), t, x), \varphi^2(\tau(t, x), t, x)) \). The map \( \psi^{-1} \) is differentiable if \( \tau \) is and \( \tau \) is differentiable if
\[
\frac{\partial \varphi^1(u, t, x)}{\partial u} \bigg|_{u=\tau(t,x)} \neq 0.
\]

But
\[
\frac{\partial \varphi^1(u, t, x)}{\partial u} \bigg|_{u=\tau(t,x)} = X_1(\varphi(\tau(t, x), t, x) = X_1(0, \varphi^2) = 1.
\]
It is easy now to check that $H_v$ is our required homotopy.

The following amplifies Theorem B.

**Theorem 4.** The space $\mathcal{F}$ is contractible. In fact there is a homotopy $G_v: \mathcal{F} \to \mathcal{F}$ such that for each $f \in \mathcal{F}$,

(a) $G_v(f)(x, t)$ is $C^\infty$ in $(v, x, t)$,
(b) $G_0(f) = \tilde{e}$,
(c) $G_1(f) = f$, and
(d) $G_v(e) = e$.

**Proof.** Let $K_v: \mathcal{F}_v \to \mathcal{F}$ be defined by $K_v(f)(t) = f(t)v + t(1-v)$. Then $K_v(f)(t)$ is $C^\infty$ in $(t, v)$, $K_0(f)(t) = t$, $K_1(f) = f$, and if $e$ is the identity $K_v(e) = e$.

Let $H_v$ be as in Lemma 2, and define $h_v = H_v(h)$ for each $h \in \mathcal{E}$. Let $\tilde{h}_v$ denote $h_v$ restricted to $I_1$. Let $\beta(t)$ be a $C^\infty$ function of $t$ such that $\beta(t) = 0$ in a neighborhood of 0, $\beta(t) = 1$ in a neighborhood of 1.

The desired homotopy $G_v: \mathcal{F} \to \mathcal{F}$ is defined as follows.

$G_v(h)(t, x) = (t', [K_{\beta}(t')(\tilde{h}_v^{-1})](x'))$

where $t'$ and $x'$ are the $t$ and $x$ components respectively of $h_v(t, x)$. The map $(t, x) \to (t', x')$ is a diffeomorphism because $h_v$ is. It is easily checked that

$(t', x') \to (t', [K_{\beta}(t')(\tilde{h}_v^{-1})](x'))$

is a diffeomorphism. Hence the composition $G_v(h)$ is also a diffeomorphism. One can further check that $G_v(h)$ satisfies all the properties demanded by the theorem.

Let $S^2$ be the unit sphere in $E^3$ with Cartesian coordinates $(x, y, z)$ and $x_0$ the South Pole. Let $e_1, e_2$ be unit tangent vectors of $S^2$ at $x_0$ in the directions of the $x$ and $y$ axes respectively. Let $\Omega_0$ be the space of diffeomorphisms of $S^2$ such that if $f \in \Omega_0$ then $f(x_0) = x_0$ and $df_{x_0}(e_i) = e_i$, $i = 1, 2$.

**Theorem 5.** The space $\Omega_0$ is contractible in the sense of Theorem 4.

**Proof.** Let $E$ be the open southern hemisphere of $S^2$. Then $E^3$ induces a natural Euclidean coordinate system on $E$ and the tangent bundle $T$ of $E$. We will use these coordinates to add points on $E$ and $T$.

Let $g: \Omega_0 \to \mathbb{R}$ be defined by

$g(f) = \sup \{ q \leq 1 \mid \text{for all } X \in T(N_q(x_0)), \text{ } \left| df(X) - X \right| < 1 \}$

where $T(N_q(x_0))$ is the unit tangent bundle of $N_q(x_0)$. Then $g$ is
lower semi-continuous. Hence by Lemma 3 let \( \varepsilon \) be a positive continuous function on \( \Omega_0 \) such that \( |df(x) - X| < 1 \) for \( X \in T(N_{\varepsilon(f)}(x_0)) \) and \( f \in \Omega_0 \).

Let \( \gamma \) be a function on \( \Omega_0 \times S^2 \), with \( \gamma(f, x) \in C^\infty \) for each \( f \) and such that \( \gamma(f, x) = 1 \) for \( x \in N_{\varepsilon(f)/2}(x_0) \) and \( \gamma(f, x) = 0 \) for \( x \) in the complement of \( N_{\varepsilon(f)}(x_0) \).

Define \( S_v : \Omega_0 \to \Omega_0 \) by

\[
S_v(f)(x) = (1 - v)f(x) + v[\gamma(f, x)x + (1 - \gamma(f, x))f(x)].
\]

Then \( S_0(f) = f \) and \( S_1(f) \) agrees with the identity on \( N_{\varepsilon(f)/2}(x_0) \).

Let \( p : S^2 - x_0 \to E^2 \) be stereographic projection using \( x_0 \) as pole with \( E^2 \) being the \( x - y \) plane of \( E^3 \). Let \( I_M^2 \) denote the square in \( E^2 \) with center \((0, 0)\) and with sides of length \( M \) which are parallel to the axes.

Let \( M \) be a positive continuous function on \( \Omega_0 \) such that \( p^{-1}(\text{exterior } I_M^2) \subseteq N_{\varepsilon(f)/2}(x_0) \). Let \( \mathcal{F} \) be the space of diffeomorphisms of \( E^2 \) which are the identity in a neighborhood of \( I_M^2 \) and outside \( I_M^2 \). Then let \( G_v : \mathcal{F} \to \mathcal{F} \) be as in Theorem 4. Let a homotopy \( F_v : \Omega_0 \to \Omega_0 \) be defined by

\[
F_v(f)(x) = b^{-1}[G_v(b^{-1})]b(x), \quad b = D_M p, \quad x \in N_{\varepsilon(f)/2}(x_0),
\]

\[
F_v(f)(x) = f(x) \quad x \in N_{\varepsilon(f)/2}(x_0).
\]

Let \( \beta(v) \) be a \( C^\infty \) function which is 0 in a neighborhood of 0 and 1 in a neighborhood of 1/2. Let \( \gamma(v) \) be a \( C^\infty \) function which is 0 in a neighborhood of 1/2 and 1 in a neighborhood of 1. Then we define our desired contraction \( T_v : \Omega_0 \to \Omega_0 \) by

\[
T_v = S_{\beta(v)} \quad 0 \leq v \leq 1/2,
\]

\[
T_v = F_{\gamma(v)} \quad 1/2 \leq v \leq 1.
\]

The following amplifies Theorem A.

**Theorem 6.** The space \( \Omega \) of all orientation preserving diffeomorphisms of \( S^2 \) has as a deformation retract the rotation group \( SO(3) \). In fact there is a homotopy \( H_v : \Omega \to \Omega \) such that for each \( f \in \Omega \)

(a) \( H_v(f)(x) \) is \( C^\infty \) in \((v, x)\),

(b) \( H_0(f) = f \),

(c) \( H_1(f) \) is a rotation of \( S^2 \), and

(d) if \( f \in SO(3) \), \( H_v(f) = f \).

**Proof.** Define \( \Omega \) as the subspace of \( \Omega \) with the property that for \( f \in \Omega \), \( e^1(f) = df_{x_0}(e_1) \) and \( e^2(f) = df_{x_0}(e_2) \) are orthonormal. We first show that \( \Omega \) is a deformation retract of \( \Omega \).

If \( f \in \Omega \), let \( v_0 \) be the unit vector perpendicular to \( e^1(f) \) in \( S_f(x_0) \) the
tangent space of $S^2$ at $f(x_0)$, and $u_0$ be $e^1(f)$ normalized. Then define

$$e^1_t(f) = (1-t)e^1(f) + tu_0,$$
$$e^2_t(f) = (1-t)e^2(f) + tv_0.$$

Let $g_t$ be the linear transformation which sends $(e^1(f), e^2(f))$ into $(e^1_t(f), e^2_t(f))$. Let $p: S^2 \to S_{f(x_0)}$ be the natural projection. By Lemma 3, choose a positive continuous function $\varepsilon$ on $\Omega$ with $\varepsilon(f) < 1$, such that for a unit tangent vector $X$ in the tangent space of $N_{\varepsilon(f)}(f(x_0))$, $d(p^{-1}g_t p)(X)$ and $X$ are independent.

Let $\gamma$ be a function on $\Omega \times S^2$ such that for each $f \in \Omega$, $\gamma(f, x)$ is $C^\infty$, is zero outside $N_{\varepsilon(f)}(f(x_0))$ and is 1 on $N_{\varepsilon(f), \rho}(f(x_0))$. Let $p$ induce an affine structure on the hemisphere of $S^2$ with center $f(x_0)$. Define $G_v: \Omega \to \Omega$ by

$$G_v(f)(x) = \gamma(f, x)p^{-1}g_v p(x) + (1 - \gamma(f, x))x.$$ 

Then $G_v$ retracts $\Omega$ onto $\Omega$.

We now define a retraction of $\bar{\Omega}$ onto $SO(3)$. For each $f \in \Omega$ let $\alpha(f) \in SO(3)$ be the rotation sending $(f(x_0), df(e_1), df(e_2))$ into $(x_0, e_1, e_2)$. Then define $K_v: \bar{\Omega} \to \bar{\Omega}$ by $K_v(f) = T_v(f \circ \alpha(f))\alpha(f)^{-1}$ where $T_v$ is as in Theorem 5.

The desired homotopy $H_v: \Omega \to \Omega$ is obtained by composing $G_v$ and $K_v$ as in the proof of the previous theorem.

References


Institute for Advanced Study