HYPERSPACES OF THE INVERSE LIMIT SPACE

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Introduction. Throughout the following $X$ will denote a metric continuum, $2^X$ the set of all nonempty closed subsets of $X$ and $C(X)$ the set of all nonempty subcontinua of $X$. It is the purpose of this paper to answer questions raised in [4] about the dimension and homological properties of $C(X)$ when $X$ is non-Peanian. In §1 $C(X)$ is shown to be acyclic in all dimensions and in §2 sufficient conditions for the finite dimensionality of $C(X)$ are obtained.

Notation. If $(U_1, \ldots, U_n)$ is a collection of subsets of a topological space $X$, then $\langle U_1, \ldots, U_n \rangle$ denotes $\{ E \subseteq 2^X \mid E \subseteq \bigcup_{i=1}^{n} U_i \}$ and $E \cap U_i \neq \emptyset$ for each $i$. If $X$ is a topological space, then the finite topology on $2^X$ is the one generated by collections of the form $\langle U_1, \ldots, U_n \rangle$ with $U_1, \ldots, U_n$ open subsets of $X$.

$C(X)$ denotes the space of all nonempty subcontinua of $X$ with the topology inherited from $2^X$ with the finite topology. If $X = \lim(X_i, f_i, I)$ where $X_i$ is a metric continuum, $f_i$ is continuous and $I$ is the set of natural numbers, then $X$ is a metric continuum. [See [1] for an explanation of this notation used in the description of the inverse limit space.] Now $C(X_i)$ is defined and we define $f_i^*: C(X_{i+1}) \to C(X_i)$ by $f_i^*(E) = (f_i^*|C(X_{i+1}))(E) = f_i(E)$, where $f_i^*: 2^{X_{i+1}} \to 2^{X_i}$ is continuous by [6, Theorem 5.10], so that $f_i^*$ is continuous. Let $C_\infty(X) = \lim (C(X_i), f_i^*, I)$, where $C_\infty(X)$ is given the relative topology inherited from the product of the $C(X_i)$'s with the product topology. Let $\pi_n: X \to X_n$ be the projection map on $X$ and $\pi_n': C_\infty(X) \to C(X_n)$ be the projection map on $C_\infty(X)$.

1. Homology of $C(X)$. First we show that $C(X)$ and $C_\infty(X)$ are homeomorphic.

Lemma 1.1. $\{ \langle U_1, \ldots, U_k \rangle \mid U_1, \ldots, U_k \text{ open in } X \}$ forms a basis for $C(X)$.

Proof. [6, Theorem 2.1].

Lemma 1.2. $\{ \pi_n^{-1}(\langle U_1, \ldots, U_k \rangle) \mid n \in I \text{ and } \langle U_1, \ldots, U_k \rangle \text{ open in } C(X_n) \}$ forms a basis for $C_\infty(X)$.

Proof. [1, Lemma 3.12, p. 218].
Lemma 1.3. \( \{ \langle \pi_n^{-1}(U_1), \ldots, \pi_n^{-1}(U_k) \rangle \} \) forms a basis for \( C(X) \).

Proof. If \( V \) is a basic open set in \( C(X) \), then \( V = \langle V^1, \ldots, V^k \rangle = \{ G \mid G \subseteq C(X), G \subseteq U_{i=1}^k V^i \text{ and } G \cap V^i \neq \emptyset \text{ for } i = 1, \ldots, k \} \). Let \( e_0 \) be the Lebesgue number of \( \langle V^1, \ldots, V^k \rangle \). Let \( \epsilon_i > 0 \) for \( i = 1, \ldots, k \) be such that there exist \( x^i \in V^i \) such that \( S_{\epsilon_i}(x^i) \subseteq V^i \) where \( S_{\epsilon_i}(x^i) \) is a spherical open set with center \( x^i \) and radius \( \epsilon_i \). Let \( \epsilon = \min \{ \epsilon_i \mid i = 0, 1, \ldots, k \} \). Now there exist \( n(e) \) and \( \eta(e) \) such that if \( A \) is a subset of \( X_n \) and \( \text{diam} (A) < \eta \), then \( \text{diam} \pi_n^{-1}(A) < e \). Cover \( G_n = \pi_n(G) \) with open sets of diameter less than \( \eta \), since \( G_n \) is compact we need only a finite number of these open sets to cover \( G_n \). Choose a finite irreducible set of such open sets and call them \( T_1, \ldots, T_m \). We have \( G_n \subseteq \bigcup_{j=1}^m T_j \) and \( T_j \cap G_n \neq \emptyset \) for \( j = 1, \ldots, m \) and \( \text{diam} (T_j) < \eta \).

So \( G \subseteq \bigcup_{j=1}^m \pi_n^{-1}(T_j) \) and \( \bigcap \pi_n^{-1}(T_j) \neq \emptyset \) for \( j = 1, \ldots, m \) and \( \text{diam} \pi_n^{-1}(T_j) < \epsilon \). Therefore \( \pi_n^{-1}(T_j) \) is contained in some \( V^i \). Let \( T_i = \bigcup \{ \pi_n^{-1}(T_j) \mid \pi_n^{-1}(T_j) \subseteq V^i \} \). Since for each \( i \) we have \( x^i_n = \pi_n(x^i) \in T_j \) and \( \text{diam} (T_j) < \eta \) we have \( x^i_n \in \pi_n^{-1}(T_j) \) and \( \text{diam} \pi_n^{-1}(T_j) < \epsilon \leq \epsilon_i \). Therefore there exists a \( \pi_n^{-1}(T_j) \subseteq V^i \) for each \( i \), so that \( T_i \neq \emptyset \). Therefore \( T_i \) is a nonnull open set of \( X \) of the form \( \pi_n^{-1}(\bigcup T_j) \subseteq V^i \) where \( \bigcup T_j \) is open in \( X_n \).

Consider \( \langle T^1, \ldots, T^k \rangle \) it is of the desired form. We must show \( G \subseteq \langle T^1, \ldots, T^k \rangle \) and \( V \) is the union of such \( \langle T^1, \ldots, T^k \rangle \)'s.

First we show \( G \subseteq \langle T^1, \ldots, T^k \rangle \). \( G \subseteq \bigcup_{j=1}^m \pi_n^{-1}(T_j) \) and \( \bigcap \pi_n^{-1}(T_j) \neq \emptyset \) for each \( j \). Since \( \bigcup_{j=1}^m \pi_n^{-1}(T_j) = \bigcup_{i=1}^k T^i \) we have \( G \subseteq \bigcup_{i=1}^k T^i \) and \( \bigcap \bigcup_{i=1}^k T^i \neq \emptyset \) for each \( i \). Therefore \( G \subseteq \langle T^1, \ldots, T^k \rangle \).

Second we show \( V = \bigcup \{ \langle T^1, \ldots, T^i \rangle \} \). Since \( G \in V \) implies \( G \subseteq \bigcup \{ \langle T^1, \ldots, T^k \rangle \} \) we have \( V \subseteq \bigcup \{ \langle T^1, \ldots, T^k \rangle \} \).

If \( A \subseteq \bigcup \{ \langle T^1, \ldots, T^i \rangle \} \) then \( A \subseteq \langle T^1, \ldots, T^i \rangle \) and so \( A \subseteq \bigcup_{p=1}^k T^p \) and \( A \cap T^p \neq \emptyset \) for each \( p \). Therefore since \( \bigcup_{p=1}^k T^p \subseteq \bigcup_{i=1}^k V^i \) we have \( A \subseteq \bigcup_{i=1}^k V^i \) and \( A \cap \bigcup_{i=1}^k \pi_n^{-1}(V^i) \neq \emptyset \) where \( \pi_n^{-1}(V^i) \subseteq V^i \) for \( i = 1, \ldots, k \). Therefore \( A \subseteq \bigcup_{i=1}^k V^i \) and \( \bigcup_{i=1}^k V^i \neq \emptyset \) for \( i = 1, \ldots, k \). Therefore \( A \in V \) and \( \bigcup \{ \langle T^1, \ldots, T^i \rangle \} \subseteq V \). Therefore \( V = \bigcup \{ \langle T^1, \ldots, T^i \rangle \} \).

Theorem 1.1. \( C(X) \) and \( C_\infty(X) \) are homeomorphic.

Proof. If \( A \subseteq C_\infty(X) \) then \( A = (A_1, A_2, A_3, \ldots) \) where \( A_i \subseteq C(X_i) \). If \( D \subseteq C(X) \) then \( D = \{ (x_1, x_2, \ldots) \mid x_i \in D_i = \pi_i(D) \} \). We define \( h: C_\infty(X) \rightarrow C(X) \) by \( h(A) = \{ (x_1, x_2, \ldots) \mid x_i \in A_i \} \). If \( h(A) = h(B) \) then \( \{ (x_1, x_2, \ldots) \mid x_i \in A_i \} = \{ (y_1, y_2, \ldots) \mid y_i \in B_i \} \). Now for any \( x_i \in A_i \), there is an \( (x_1, x_2, \ldots, x_i, \ldots) \) equal to \( (y_1, \ldots, y_i, \ldots) \) and hence \( x_i = y_i \) so that \( x_i \in B_i \). Therefore \( A \subseteq B_i \) and in the same
way $B_i \subseteq A_i$ so that $A_i = B_i$ for each $i$. Therefore $A = B$ and $h$ is 1-1. If $B \in C(X)$ then $B = \{(x_1, x_2, \ldots) | x_i \in B_i\} = h((B_1, B_2, \ldots))$ so that $h$ is onto.

By Lemma 1.3 $\langle \pi_n^{-1}(U_1), \ldots, \pi_n^{-1}(U_k) \rangle$ is a basic open set so to show $h$ is continuous we will show that $h^{-1}(\langle \pi_n^{-1}(U_1), \ldots, \pi_n^{-1}(U_k) \rangle)$ is an open set in $C_\infty(X)$.

$$h^{-1}(\langle \pi_n^{-1}(U_1), \ldots, \pi_n^{-1}(U_k) \rangle)$$

$$= \{ A \in C_\infty(X) | h(A) \in \langle \pi_n^{-1}(U_1), \ldots, \pi_n^{-1}(U_k) \rangle \}$$

$$= \left\{ A \in C_\infty(X) \mid h(A) \subset \bigcup_{i=1}^{k} \pi_n^{-1}(U_i) \text{ and } h(A) \cap \pi_n^{-1}(U_i) \neq \emptyset \right\}$$

$$= \left\{ A \in C_\infty(X) \mid \pi_n h(A) \subset \bigcup_{i=1}^{k} U_i \text{ and } \pi_n h(A) \cap U_i \neq \emptyset \right\}$$

$$= \left\{ A \in C_\infty(X) \mid \pi_n \left( \{ (x_1, \ldots) \mid x_i \in A_i \} \right) \in \langle U_1, \ldots, U_k \rangle \right\}$$

$$= \left\{ A \in C_\infty(X) \mid \{ x_n \mid x_n \in A_n \} \in \langle U_1, \ldots, U_k \rangle \right\}$$

$$= \left\{ A \in C_\infty(X) \mid A \in \langle (U_1, \ldots, U_k) \rangle \right\}$$

$$= \pi_n^{-1}(\langle U_1, \ldots, U_k \rangle) \text{ open in } C_\infty(X).$$

**Property 3.2.** For $\epsilon > 0$, there exists $d(\epsilon) > 0$ such that if $a, b \in X$, $\text{dist}(a, b) < d(\epsilon)$ and $a, b \in A \in C(X)$, then there exists $B$ such that $b \in B \in C(X)$ with the Hausdorff distance from $A$ to $B$ less than $\epsilon$.

**Theorem 1.2.** If $X$ is a metric continuum then $C(X)$ is acyclic in all dimensions.

**Proof.** By [2, p. 183] $X = \lim (X_i, f_i, I)$ where $X_i$ is a polyhedron, $f_i$ is continuous and onto, $I$ is the set of natural numbers. If $Y$ has property 3.2 by [4, Theorem 3.4] the Vietoris groups $V_n(C(Y)) = 0$. Now a polyhedron $P$ has property 3.2 so $V_n(C(P)) = 0$. By [5, Theorem 26.1] for a compact metric space $Y$, $V_n(Y) = H_n(Y)$ where the Vietoris groups $V_n$ and the Čech groups $H_n$ are taken over a discrete group. So using the above, Theorem 1.1 and the continuity of Čech theory we have the following: $V_n(C(X)) = H_n(C(X)) = H_n(C_\infty(X)) = H_n(\lim (C(X_i), f'_i, I)) = \lim (H_n(C(X_i)), f'_i, I) = \lim (O_i, f'_i, I) = 0$ where $O_i = 0$.

2. **Dimension of $C(X).$** Kelley leaves as an open question the dimension of $C(X)$ when $X$ is not locally connected. If $X$ is a metric continuum of dimension $n$, then $X = \lim (X_i, f_i, I)$ where $X_i$ is a polyhedron of dimension $n$. If in addition $\dim C(X_i) \leq k$ for all $i$ we shall say $X$ has property $k$ (with respect to $\lim (X_i, f_i, I)$).
Theorem 2.1. If \( \dim (X) = 1 \) and \( X \) has property \( k \), then \( \dim C(X) < \infty \).

Proof. By [4, Theorem 5.4] (if \( X \) is Peanian then \( \dim C(X) < \infty \) if and only if \( X \) is a linear graph) we have since \( \dim C(X_i) \leq k \) for all \( i \) that \( k < \infty \). Therefore \( \dim C(X) = \dim C_\omega(X) \leq k < \infty \).

Example 2.1. Let \( X \) be the dyadic solenoid, then since \( X = \lim (X_i, f_i, I) \) where \( X_i = S^1 \) and \( f_i(z) = z^2 \), we have \( \dim C(X_i) = 2 \) for each \( i \), hence the \( \dim C(X) = \dim C_\omega(X) \leq 2 \).

Example 2.2. To see the need of imposing property \( k \) in Theorem 2.1 consider the following: let \( X_i \) be the union of \( 2^i \) straight line segments \( A^i_0, \ldots, A^i_{2^i-1} \) where \( A^i_j \) for \( j = 0, \ldots, 2^i-1 \) is from \((0,0)\) to \((1,j\pi/2^i)\) in the plane (polar coordinates). Let \( f_i: X_{i+1} \rightarrow X_i \) be the identity map on \( A^i_{j+1}, A^i_{j+1}, A^i_{j+1}, \ldots, A^i_{2^i-1} \) where \( f_i(A^i_j) = A^i_{j/2} \) for \( j = 0, 2, \ldots, 2^i-1 \), and \( f_i \) maps \( A^i_j \) linearly onto \( A^i_{(j-1)/2} \) keeping the origin fixed for \( j = 1, 3, \ldots, 2^i-1 \). Then \( X \) is a Cantor set of arcs meeting at a single point \( (\vec{x}_i) = \vec{x} \) where \( \vec{x}_i = (0,0) \) for each \( i \). Now the \( \dim C(X_i) = 2 + \sum \text{order } x_i \vec{x}^2 \) (order \( x_i - 2 \)) = \( 2 + \text{order } \vec{x} - 2 \) = \( \text{order } \vec{x} = 2^i \), so that \( X \) fails to have property \( k \). Further \( \dim C(X) \) is infinite since the order \( \vec{x} = \infty \).

Bibliography