MEROMORPHIC FUNCTIONS OF FINITE ORDER

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1. Introduction. Let \( f(z) \) be an entire function of finite order \( \rho \). If the genus \( p \) of the canonical product (c.p.) formed with zeros of \( f(z) \) is equal to the genus of \( f(z) \), then it is known\(^1\) that \([9; 10]\)

\[
\lim inf_{r \to \infty} \frac{\log M(r)}{n(r, 0)\phi(r)} = 0
\]

where \( \phi(r) \) is any positive nondecreasing function such that

\[
\int_1^\infty \frac{dt}{t\phi(t)}
\]

is convergent. Another proof of this theorem is given by Boas [2] where he assumes only that \( \phi(r) \) is positive and (1) exists. Following the method of Boas, Shanker Hari has shown in a paper under publication that if \( f(z) \) is a c.p. of order and genus \( \rho \) then

\[
\lim inf_{r \to \infty} \frac{\log M(r)}{N(r, 0)\phi(r)} = 0.
\]

In this paper we prove a more general theorem by the method given in \([9; 10]\).

Let \( G, p_1, p_2 \) denote respectively the genus of a meromorphic function \( f(z) \), the genus of the c.p. \( P_1 \) formed with zeros of \( f \) and the genus of the c.p. \( P_2 \) formed with poles of \( f \). Write\(^2\) \( S = \max(p_1, p_2) \), \( n(t) = n(t, f - 0) + n(t, f - \infty) \), \( n_1(t) = n(t, P_1 - 0) + n(t, P_2 - 0) \). Let \( k \) denote any constant \( \geq 1 \), \( \phi(x) \) any positive nondecreasing function such that (1) exists, \( (a_i(z)) \), \( i = 0, 1, \cdots, l \) any meromorphic functions (including rational functions or finite constants) such that

\[
T(r, a_i(z)) = o(T(r, f)), \quad i = 0, 1, \cdots, l
\]

and \( g_0(z) = g(z) \), \( g_1(z) \), \( g_2(z) \) any meromorphic functions (including rational functions or constants, finite or infinite) such that

\[
T(r, g_i(z)) = o(T(r, f)), \quad i = 0, 1, 2.
\]

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\(^1\) For notations see 7 [7; 8].

\(^2\) The function \( n(r, f - g) \) denotes the number of zeros of \( f - g \) in \( |z| \leq r \). If \( g \) is infinite constant, then \( N(r, 1/(f - g)) = N(r, f) \).
Write
\[
\psi(f) = \sum_{i=0}^{l} a_i(z)f^{(i)}(z), \quad \psi(f - g_1) = \sum_{i=0}^{l} a_i(z)\{f^{(i)}(z) - g_1^{(i)}(z)\}
\]

when \( g_1 \) is not infinite constant; we suppose that \( (a_i) \) are so chosen that neither \( \psi(f) \) nor \( \psi(f - g_1) \) reduces identically to a constant.

**Theorem 1.** Let \( f(z) \) be a meromorphic function of integer order \( \rho \).

(a) If \( G = S \), then

\[
\lim\inf_{r \to \infty} \frac{T(kr, f)}{(n(r, 0) + n(r, \infty))\phi(r)} = 0,
\]

(b) If \( G > S \), then

\[
\lim\inf_{r \to \infty} \frac{T(kr, f)}{(n(r, 0) + n(r, \infty))\phi(r)} = 0.
\]

**Theorem 2.** Let \( f(z) \) be a meromorphic function of noninteger order \( \rho > 0 \). Then

(a)

\[
\lim\inf_{r \to \infty} \frac{T(kr, f)}{(n(r, f - g_1) + n(r, f - g_2))} \leq k^{1+S}B(S),
\]

\[
\lim\inf_{r \to \infty} \frac{T(kr, f)}{(N(r, 1/(f - g_1)) + N(r, 1/(f - g_2))} \leq k^{1+S}(S + 1)B(S)
\]

where

\[
S = [\rho], \quad A(S) = 2 \quad \text{if } S = 0,
\]

\[
= 2e(2 + \log S)(S + 1)^2/S \quad \text{if } S \geq 1,
\]

and

\[
B(S) = A(S)/(\rho - S)(1 + S - \rho)
\]

and \( g_1, g_2 \) are any two distinct functions satisfying (3).

**Corollary.**

\[
\lim\inf_{r \to \infty} \frac{T(r, f)}{n(r, f - g)} \leq 2B(S),
\]
(10) \[ \liminf_{r \to \infty} \frac{T(r, f)}{N(r, 1/(f - g))} \leq 2(S + 1)B(S) \]

except possibly for one \( g \) in each relation (9), (10).

(b) If (10) does not hold for \( g = g_1 \), then for all \( r > r_0 \),

\[ T(r, f) < \left\{ \frac{2(S + 1)B(S)}{2(S + 1)B(S) - 1} \right\} \left\{ N(r, f) + \overline{N}(r, 1/(\psi(f - g) - 1)) \right\} \]

if \( g_1 \neq \infty \) and

\[ T(r, f) < \left\{ \frac{2(S + 1)B(S)}{2(S + 1)B(S) - 1} \right\} \left\{ N(r, f) + \overline{N}(r, 1/(\psi(f) - 1)) \right\} \]

if \( g_1 = \infty \).

(c) If (9) does not hold for \( g = g_1 \), then we have for a sequence of values of \( r \uparrow \infty \),

\[ T(r, f) < \left\{ \frac{2\rho B(S)}{2\rho B(S) - 1} \right\} \left\{ N(r, f) + \overline{N}(r, 1/(\psi(f - g) - 1)) \right\} \]

if \( g_1 \neq \infty \) and

\[ T(r, f) < \left\{ \frac{2\rho B(S)}{2\rho B(S) - 1} \right\} \left\{ N(r, f) + \overline{N}(r, 1/(\psi(f) - 1)) \right\} \]

if \( g_1 = \infty \).

**Theorem 3.** Let \( f(z) \) be an entire function of order \( \rho \) and let, when \( \rho > 0 \), \( h(\rho) = (\rho + (1 + \rho^2)^{1/2})(1 + (1 + \rho^2)^{1/2})^{\rho}/\rho^\rho \). If \( \rho > 0 \), then

\[ \liminf_{r \to \infty} \frac{\log M(kr)}{n(r, f - g)} \leq \frac{2k^\rho h(\rho)}{\rho} \],

\[ \liminf_{r \to \infty} \frac{\log M(kr)}{\overline{N}(r, 1/(f - g))} \leq 2k^\rho h(\rho) \]

for every entire function \( g(z) \) (including a polynomial or a finite constant) satisfying (3), with one possible exception. If \( \rho = 0 \), then

\[ \liminf_{r \to \infty} \frac{\log M(kr)}{\overline{N}(r, 1/(f - g))} = 1 \]

for every entire function \( g(z) \) such that \( M(r, g) = o(M(r, f)) \).

**Theorem 4.** Let \( f(z) \) be a meromorphic function of order \( \rho \). If \( \rho > 0 \), then
\[ T(kr,f) \leq \frac{3k^\rho}{\rho}, \]
\[ T(kr,f) \leq 3k^\rho \]
for every meromorphic function \( g(z) \) satisfying (3), with two possible exceptions. If \( \rho = 0 \) then
\[ \liminf_{r \to \infty} \frac{T(kr,f)}{N(r,1/(f-g))} \leq 1 \]
where \( g_1, g_2 \) are any two distinct functions satisfying (3).

2. Remarks. (i) Let \( \delta(\alpha) \) denote Nevanlinna defect. If \( \sum \delta(\alpha) < 2 \), then \( G = S \) and (4) holds. If \( \sum \delta(\alpha) = 2 \), then (4) may or may not hold. (See [13, pp. 590–591] for an entire function of which the genus is greater than the genus of its c.p. and for which \( \sum \delta(\alpha) = 2 \) and (4) holds.) Conversely if (4) holds, then \( \sum \delta(\alpha) \) may have its maximum value \( 2 \) or \( \sum \delta(\alpha) < 2 \) \( (f(z) = \sin z) \). If (4) does not hold then \( G > S \), \( \delta(0) = \delta(\infty) = 1 \).

(ii) For entire functions of order \( \rho, 0 \leq \rho < 1 \), it is known that
\[ \liminf_{r \to \infty} \frac{\log M(r)}{N(r,a)} \leq \frac{1}{1 - \rho} \]
and for meromorphic functions of order \( \rho, 0 \leq \rho < 1 \), it is known that
\[ \liminf_{r \to \infty} \frac{T(r)}{N(r,a) + N(r,b)} \leq \frac{1}{1 - \rho} \]
More precise results in this direction have been given by Edrei and Fuchs [4].

(iii) For entire functions of noninteger order \( \rho > 0 \), it is known that
\[ L(f,a) = \liminf_{r \to \infty} \frac{\log M(r)}{n(r,f-a)} < \infty \]
for every finite \( a \). A sharp upper bound for \( L(f,a) \) when \( \rho > 1 \), is not known but by (15) \( L(f,a) \leq 2h(\rho)/\rho \) except possibly for one value of \( a \). This exceptional value may exist. Consider [7, pp. 18–19]
\[ f(z) = \prod_{n=2}^{\infty} E\left( \frac{z}{a_n}, \rho \right), \quad a_n = -n^{1/p}, \quad p < \rho < p + 1. \]
f(\( z) \) is an entire function of order \( \rho \) and
\[ \log M(r)/n(r, 0) \sim \frac{A}{\sin \pi (\rho - p)} \]

which can be made greater than \(2h(\rho)/\rho\) by choosing \(\rho - p\) sufficiently small.

(iv) If (15), (16), (18) or (19) do not hold for \(g = g_1\) then we can obtain relations similar to (11)–(14). For instance if (15) or (16) does not hold when \(g = g_1\), then

\[
\lim_{r \to \infty} \frac{T(r, f)}{N(r, 1/(\psi(f - g_1) - 1))} \leq 2.
\]

3. Theorem 1.

**Lemma 1.** If \(f(z) = z^v \exp(Q(z))P_1(z)/P_2(z)\) is a proper meromorphic function\(^3\) of order \(\rho \geq 0\) such that \(G = S\), then for all \(r > r_0\)

\[ T(kr, f) \leq A(S)k^{1+S}J(r, S) \]

where\(^4\)

\[
A(S) = \begin{cases} 2 & \text{if } S = 0, \\ 2e(2 + \log S)(S + 1)^2/S & \text{if } S \geq 1, \end{cases}
\]

and

\[
J(r, S) = r^{1+S} \int_0^\infty \frac{n_1(t)dt}{t^{1+S}(t + r)}.
\]

**Proof.** We have

\[ T(kr, f) = O(r^S + \log r) + \log M(kr, P_1) + \log M(kr, P_2) \]

and when \(p_1 \geq 1\) [8, pp. 225–226]

\[
\log M(kr, P_1) \leq (2 + 1/p_1)e(2 + \log p_1)(1 + p_1)k^{1+p_1}r^{1+p_1} \int_0^\infty \frac{n(t, P_1 - 0)dt}{t^{1+p_1}(t + kr)}.
\]

Hence, when \(p_1 = p_2 \geq 1\),

\[ T(kr, f) \leq \frac{2e(2 + \log S)(1 + S)^2}{S} k^{1+S}J(r, S). \]

If \(p_1 = p_2 = 0\), then

\[ T(kr, f) < O(\log r) + kr \int_0^\infty \frac{n_1(t)dt}{t(t + kr)}. \]

\(^3\) \(f\) is not a rational function.

\(^4\) We can take \(A(0) = 1 + c\), where \(c\) is any positive number.
If $p_1 > p_2$ then $\log M(kr, P_2) = o(r^{1+p_2}) = o(r^{p_1})$ and a similar result if $p_1 < p_2$. Hence the lemma is proved.

**Lemma 2.** If $f(z)$ is a proper meromorphic function of finite order and
\[
\psi(f) = \psi(z) = \sum_{i=0}^{1} a_i(z)f^{(i)}(z)
\]
then
\[
(21) \quad (1 + o(1))T(r, f) < N(r, f) + N\left(r, \frac{1}{\psi(f) - 1}\right) + N\left(r, \frac{1}{f}\right).
\]

This lemma is substantially contained in Theorem 7 of [5].

**Lemma 3.** If $f(z) = z^p \exp(Q(z))P_1(z)/P_2(z)$ and $G > S$ then $Q(z) = a_pz^p + \cdots$, $|a_p| = A \neq 0$ and
\[
(i) \quad T(r, f) \sim Ar^p,
(ii) \quad T(r, f') \sim Ar^p,
(iii) \quad T(r, \psi) = T(r, \sum a_i f^{(i)}) \sim Ar^p.
\]

**Proof.** Write $f = b_0 \exp(Q)$ where $b_0 = z^p P_1 P_2^{-1}$,
\[
T(r, f) \leq T(r, \exp Q) + T(r, b_0),
\]
and since $T(r, \exp Q) \sim Ar^p$, $T(r, b_0) = o(r^p)$, (i) follows.

\[
T(r, f') = e^Q\left\{kz^{k-1}P_1 P_2^{-1} + z^k Q' P_1 P_2^{-1} + z^k P_1' P_2^{-1} - z^k P_1 P_2^{-2} P_2'\right\}
\]
\[
eq e^Qb_1 \quad \text{(say)}.
\]
Then $T(r, b_1) = o(r^p)$ and so $T(r, f') \sim Ar^p$.

(iii) $f'' = (\exp Q)b_2$ where $T(r, b_2) = o(r^p)$ and so on. Hence
\[
T(r, \psi(f)) = T(r, (\exp Q) \sum a_i b_i) \leq T(r, \exp Q) + T(r, \sum a_i b_i) \leq (A + o(1))r^p + \sum T(r, a_i) + \sum T(r, b_i) = (A + o(1))r^p.
\]

Also
\[
\exp Q = \psi(f)/\sum a_i b_i, \quad T(r, \exp Q) \leq T(r, \psi(f)) + o(r^p)
\]
and hence $T(r, \psi(f)) \sim Ar^p$.

(a) **Proof of** 5 (4). Since $\rho \geq 1$ and $G = S$, $n(x)$ tends to infinity with

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5 (4) can also be proved by the method given in [12, pp. 696–697].
Further \( T(kr, f) \sim T(kr, f^{2v}) \) where \( v \) is any integer and so we may suppose \( f(0) \neq 0, \neq 0 \). Then \( f(z) = \exp(Q(z))P_1(z)/P_2(z) \). Since \( G = S \), we have \( S = \rho \) or \( \rho - 1 \). Consider first when \( S = \rho \). If \( n(x) = O(x^S) \) we have

\[
J(r, S) < c_2 n^S \int_0^r \frac{n(x)dx}{x^{1+S}}
\]

and hence from Lemma 1 of [9, p. 24] we get

\[
\liminf_{r \to \infty} J(r, S)/n(r)\phi(r) = 0.
\]

If \( \limsup_{x \to \infty} n(x)/x^S = \infty \), we write \( y = \log x \) and \( \log(n(x)/x^S) = \log \mu(x) = \psi(y) \). Then \( \limsup_{y \to \infty} \psi(y) = \infty \), and since \( f(z) \) is of order \( \rho = S \), \( \limsup_{y \to \infty} \psi(y)/y = 0 \). Apply Lemma 2 of [9, p. 25]. There exists a sequence \( r_n \uparrow \infty \) for which

\[
\frac{\log \mu(x)}{\log x} < \frac{\log \mu(r_n)}{\log r_n}, \quad x > r_n.
\]

Hence

\[
J(r_n, S) < c_2 n(r_n) \log r_n.
\]

Consider now \( S = \rho - 1 \). Given \( \phi(x) \) we can find [10] a function \( \phi_1(x) \) such that for all large \( x \): (i) \( \phi_1(x) \leq \phi(x) \); (ii) \( \phi_1(x)/x^\alpha \) is nonincreasing, \( \alpha \) being a positive number less than unity; (iii) \( \int_a^\infty dt/\theta_1(t) \) is convergent. If now

\[
\limsup_{x \to \infty} n(x)\phi_1(x)/x^{1+S} > 0,
\]

then for a sequence \( R_n \uparrow \infty \), \( J(R_n, S) = o(n(R_n)\phi_1(R_n)) \). If \( n(x)\phi_1(x) = o(x^{1+S}) \) then we choose in Lemma 3 of [10, p. 184] \( 0 < \beta < 1 - \alpha \), \( \theta(x) = x^\beta \), \( \Psi(x) = n(x)\phi_1(x)/x^{S+1-\beta} \) and obtain for \( x_n \uparrow \infty \), \( J(x_n, S) = o(\phi(x_n)n(x_n)) \) and (2) is proved.

Since \( n(r, a) \leq N(2r, a)/\log 2 \), (5) follows from (4).

(b) PROOF OF (6). By hypothesis

\[
N(r, f) = o(r^\rho), \quad N(r, 1/f) = o(r^\rho),
\]

and so from (21)

\[
(1 + o(1))T(r, f) < \frac{N(r, 1/(\psi(f) - 1))}{T(r, \psi(f)) + O(1)}
\]

and so (6) follows from Lemma 3(i) and (iii).
4. Theorems 2, 3, 4.

**Lemma 4.** If \( f(z) \) is a meromorphic function of order \( \rho > 0 \), then

\[
I_1(r) = \int_1^r \frac{T(kt, f)}{t} \, dt < \frac{k^\rho}{\rho} (1 + \epsilon) T(r, f)
\]

for a sequence of values of \( r \) tending to infinity.

**Lemma 5.** If \( f(z) \) is an entire function of order \( \rho > 0 \), then

\[
I_2(r) = \int_1^r \frac{\log M(kt)}{t} \, dt < \frac{k^\rho}{h(\rho)} (1 + \epsilon) T(r, f)
\]

where \( h(\rho) \) has been defined in the statement of Theorem 3, for a sequence of values of \( r \) tending to infinity.

The proof of Lemma 4 is straightforward (cf. [14, p. 321]) and the proof of Lemma 5 depends on the relation

\[
\log M(r) \leq \frac{h + 1}{h - 1} T(hr), \quad h > 1.
\]

We obtain for a sequence of values of \( r \to \infty \),

\[
I_2(r) \leq \frac{h + 1}{h - 1} \frac{h^\rho k^\rho}{\rho} (1 + \epsilon) T(r).
\]

Choose \( h = \frac{1}{1 + \sqrt{1 + \rho^2}} \).

(a) **Proof of (7).** We have \( S = [\rho] \)

\[
T(kr, f) < A(S)k^{1+\delta} J(r, S), \quad r > r_0.
\]

We choose in Lemma 3 of [10, p. 184] (see also [12, p. 69]) \( \Psi(x) = n_1(x)/x^{\varepsilon-\varepsilon} \) and obtain that for \( r = r_n \uparrow \infty \)

\[
T(kr_n, f) < B(S)(1 + \epsilon') k^{1+\delta} n(r_n).
\]

Let \( H(z) = (f(z) - g_1(z))/(f(z) - g_2(z)) \), \( g_1 \neq \infty \), \( g_2 \neq \infty \). Then \( T(r, H) \sim T(r, f) \). Also for a sequence \( R_n \uparrow \infty \),

\[
T(kR_n, H) < B(S)(1 + \epsilon') k^{1+\delta} \left\{ n(R_n, H - 0) + n(R_n, H - \infty) \right\}.
\]

Now

\[
n(R_n, H - 0) + n(R_n, H - \infty) < \sum_{i=1}^2 n(R_n, f - g_i) + \sum_{i=1}^2 n(R_n, g_i - \infty).
\]

Suppose now \( k > 1 \). Then

\[
n(R_n, g_i - \infty) < N(kR_n, g_i) / \log k = o(T(kR_n, f)).
\]
Hence for \( n > n_1 \)

\[
T(kR_n, f) < B(S)(1 + \epsilon')k^{1+s}\sum_{i=1}^{2} n(R_n, f - g_i).
\]

If we take \( H(z) = f(z) - g_2(z), g_2 \not\equiv \infty \), we get for another \( \{R_n\} \uparrow \infty \),

\[
T(kR_n, f) < B(S)(1 + \epsilon')k^{1+s}\{ n(R_n, f - \infty) + n(R_n, f - g_2) \}
\]

and so from (22) and (23)

\[
\liminf_{r \to \infty} T(kr, f) \left/ \sum_{1}^{2} n(r, f - g_i) \right. \leq B(S)k^{1+s}
\]

and (7) is proved when \( k > 1 \). Further \( k \) can be chosen arbitrarily near to 1 in the relation

\[
\liminf_{r \to \infty} T(r,f) \left/ \sum_{1}^{2} n(r, f - g_i) \right. \leq B(S)k^{1+s}
\]

and so (7) is true when \( k \geq 1 \).

**Proof of (8).** Write

\[
N_1(t) = \int_{0}^{r} \left\{ n_1(t)/t \right\} dt; \quad N(t) = N(t, f) + N(t, 1/f).
\]

Then

\[
J(r, S) = \int_{0}^{\infty} \frac{r^{1+s}n_1(t)dt}{t^{1+s}(t + r)} < (S + 1) \left\{ r^s \int_{0}^{r} \frac{N_1(t)dt}{t^{1+s}} + r^{1+s} \int_{r}^{\infty} \frac{N_1(t)dt}{t^{2+s}} \right\}
\]

\[
< (S + 1)(1 + \epsilon) \left\{ r^s \int_{0}^{r} \frac{[N(t)]dt}{t^{1+s}} + r^{1+s} \int_{r}^{\infty} \frac{[N(t)]dt}{t^{s+2}} \right\}
\]

if \( r > x_1(\epsilon) \). Hence by Lemma 3 of [10, p. 184] we get

\[
\liminf_{r \to \infty} \frac{T(kr,f)}{N(r)} \leq B(S)(1 + S)k^{1+s}
\]

and the rest of the argument is the same as for (7). Hence (8) is proved. The Corollary follows from (7) and (8).

(b) If (10) is false for \( g = g_1 \not\equiv \infty \), then

\[
\liminf_{r \to \infty} \frac{T(r,f)}{N(r, 1/(f - g_1))} = \beta > 2(S + 1)B(S).
\]
Hence for all \( r > r_0(\epsilon) \),
\[
T(r, f) > (\beta - \epsilon)N(r, 1/(f - g_1))
\]
We apply Lemma 2 to \( f - g_1 \) and obtain
\[
\{1 + o(1)\} T(r, f - g_1) \leq N(r, f - g_1) + \frac{1}{\psi(f - g_1) - 1} + N(r, \frac{1}{f - g_1}),
\]
and for \( r > r_1(\epsilon) \)
\[
\left(1 - \frac{1}{\beta - \epsilon} - \epsilon\right) T(r, f) \leq N(r, f) + \frac{1}{\psi(f - g_1) - 1}.
\]
Hence for \( r > r_2 \)
\[
T(r, f) < \left\{\frac{2(S + 1)B(S)}{2(S + 1)B(S) - 1}\right\} \left\{N(r, f) + \frac{1}{\psi(f - g_1) - 1}\right\}.
\]
The proof of (12) is similar.
(c) We omit the proofs of (13)–(14) which can be proved with the help of Lemma 4.
(d) The first part of Theorem 3 follows from Lemma 5 and (17) follows from a result of Boas [1, pp. 6–7; 3, p. 48]. We omit the proofs of Theorems 3 and 4.

5. Example. We show that (4), (5) are best possible in the sense that \( k \) cannot be replaced by a function \( \alpha(r) \) tending to infinity with \( r \).
We prove that given \( \alpha(r) \to \infty \) with \( r \), there is a function \( \phi(x) \) such that (1) exists, and an entire function (c.p.) \( f(z) \) of integer order \( \rho \) and genus \( \rho - 1 \) such that
\[
T(r, f) \to \liminf_{r \to \infty} \frac{T(r\alpha(r), f)}{N(\alpha, 0)\phi(r)} = \liminf_{r \to \infty} \frac{T(r\alpha(r), f)}{n(\alpha, 0)\phi(r)} \geq \frac{1}{l(1 + 2^{1/2})^2} \text{ if } \rho = 1
\]
\[
= \infty \text{ if } \rho > 1.
\]
(a) We suppose that \( \alpha(r) \) satisfies the following:
(i) \( \alpha(r) > 0 \) for \( r \geq x_0 > 0 \) and tends to infinity with \( r \).
(ii) \( \alpha'(r) > 0 \) and \( r\alpha'(r) \) \( \to \) for \( r \geq x_0 \).
(iii) \( r^2\alpha'(r)/\alpha^2(r) \) is strictly increasing for \( r \geq x_0 \). For instance we can take \( \alpha(r) = l_k r \), \( k = 1, 2, \ldots \), \( (l_1r = \log r; l_k(r) = \log (l_{k-1}r)) \). A function of slower growth and satisfying (i)–(iii) is given by the functional equation \( \alpha(e^x) = e\alpha(x) \).
Let
\[
\phi(x) = \frac{\alpha^2(x)}{x\alpha'(x)} \quad x \geq x_0,
\]
\[
= \phi(x_0) \quad 0 \leq x < x_0,
\]
\[ n(x, 0) = \left[ \frac{x}{\phi(x)} \right] \quad \quad \quad \quad \quad \quad x \geq x_0, \]
\[ = 0 \quad \quad \quad \quad \quad \quad \quad \quad 0 \leq x < x_0. \]

From (ii), (iii) it is seen that \( \phi(x) \) and \( \frac{x}{\phi(x)} \) \( \uparrow \) with \( x \). Further

\[ \int_0^X \frac{n(t, 0)}{t^2} \, dt < \int_{x_0}^X \frac{dt}{t \phi(t)} \rightarrow \frac{1}{\alpha(x_0)} \]
as \( X \rightarrow \infty \). Hence (1) is convergent. Write \( R = \rho \alpha(r) \). Then

\[ \alpha(R) - \alpha(r) = \int_r^R \alpha'(t) \, dt = o(\alpha(R)). \]

Hence \( \alpha(R) \sim \alpha(r) \) and so \( \alpha(r) \) is "slowly increasing" function [6] and

\[ \alpha(r) = o(r^\delta), \quad \delta > 0, \quad \int_{x_0}^r \frac{dt}{\alpha(t)} \sim \frac{r}{\alpha(r)}. \]

From (ii) we have \( \alpha(r) = o(\phi(r)) \). Let

\[ f(z) = \prod_{1}^{\infty} \left( 1 + \frac{z}{r_n} \right) \]

where \( r_n > 0, \ n = 1, 2, \ldots, \) and are given by \( n(x, 0) \). (The first few zeros may be equal; cf. [8, p. 228].) From (26) we see that the series \( \sum 1/r_n \) is convergent and \( f(z) \) is an entire function (c.p.) of genus \( p = 0 \). Now

\[ \log M(r) < r \int_0^r \frac{n(t, 0)}{t} \, dt + r \int_r^\infty \frac{n(t, 0)}{t^2} \, dt \leq \left\{ 1 + o(1) \right\} \frac{r}{\alpha(r)}, \]

\[ \log M(r) > r \int_r^\infty \frac{n(t, 0) \, dt}{t(t + r)} \]
\[ > r \int_r^\infty \left\{ \frac{t}{\phi(t)} - 1 \right\} \frac{dt}{t(t + r)} = \frac{r}{\alpha(r)} \left\{ 1 + o(1) \right\}. \]

Hence

\[ \log M(r) \sim \frac{r}{\alpha(r)}, \quad \log M(R) \sim r, \quad n(r, 0) \phi(r) \sim r \sim N(r, 0) \phi(r), \]

\[ T(R) \geq \frac{\beta - 1}{\beta + 1} \log M \left( \frac{R}{\beta} \right), \quad \beta > 1, \]

\[ \sim \frac{\beta - 1}{\beta + 1} \frac{R}{\beta \alpha(r)} = \frac{\beta - 1}{\beta + 1} \frac{r}{\beta}. \]

Choose \( \beta = 1 + 2^{1/2} \) and (25) follows when \( \rho = 1 \).
(b) If $\rho > 1$ then let

$$f(z) = \prod_{i=1}^{\infty} \left(1 + \frac{z^p}{r_i}\right)$$

where $r_i$ have been defined in (a). $f(z)$ is an entire function (c.p.) of order $\rho$ and genus $\rho - 1$. We have for all $r > r_0$

$$\log M(kr, f) < c_n(r, f - 0)\phi(r)/\alpha(r),$$

$$\log M(R, f) \geq \frac{1}{\rho} (1 + o(1))(\alpha(r)^{\rho - 1}n(r, f - 0)\phi(r),$$

$$T(R, f) > c_4(\alpha(r))^{\rho - 1}n(r, f - 0)\phi(r),$$

$$T(R, f) > c_4(\alpha(r))^{\rho - 1}N(r, 1/f)\phi(r)$$

and (25) is proved for $\rho > 1$.

REFERENCES


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