ON THE ABSOLUTE HARMONIC SUMMABILITY OF A SERIES RELATED TO A FOURIER SERIES

O. P. VARSHNEY

1.1. Let \( \sum a_n \) be a given infinite series with the sequence of partial sums \( \{s_n\} \). Let the sequence \( \{t_n\} \) be defined by

\[
t_n = \frac{(n + 1)^{-1}s_0 + n^{-1}s_1 + \cdots + 1\cdot s_n}{P_n},
\]

\[
\left( P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n + 1} \right).
\]

The series \( \sum a_n \) is defined to be summable by harmonic means if the sequence \( \{t_n\} \) tends to a limit as \( n \to \infty \) [4]. If the series \( \sum |t_n - t_{n-1}| \) is convergent, we say that the series is absolutely harmonic summable. It is known that the method of summability is absolutely regular and implies absolute Cesàro summability of every positive order [2].

1.2. Let \( f(t) \) be a periodic function, with period \( 2\pi \), and integrable \( (L) \) over \( (-\pi, \pi) \). We assume without any loss of generality that the Fourier series of \( f(t) \) is given by

\[
\sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) = \sum_{n=1}^{\infty} A_n(t),
\]

and that \( \int_{-\pi}^{\pi} f(t) dt = 0 \). We write

\[
\phi(t) = \frac{1}{2} \left\{ f(x + t) + f(x - t) \right\}.
\]

Mohanty [3] has considered the absolute Riesz summability of the series

\[
\sum_{n=1}^{\infty} A_n(t)/\log (n + 1).
\]

2. In this paper we establish the following theorem:

**Theorem.** If \( \phi(t) \) is of bounded variation in \( (0, \pi) \) then the series (1.22) is absolutely summable by harmonic means.

We require the following lemmas for the proof of our theorem:

Received by the editors December 10, 1958.

784
**Lemma 1** [5]. Uniformly for $0 < t < \pi$

$$\left| \sum_{m}^{n} \frac{\sin vt}{v} \right| \leq K$$

where $m$ and $n$ are any positive integers.

**Lemma 2** [1]. If $0 < t < \pi$, then

$$\left| \sum_{k=0}^{m} \frac{\cos(k+1)t}{k+1} \right| = O\left(1 + \log \frac{1}{t}\right).$$

With the help of Lemmas 1 and 2, we may easily deduce

**Lemma 3.** If $0 < t < \pi$, then for all positive integers $m$ and $m'$

$$\sum_{k=m}^{m'} \frac{\sin(n-k)t}{k+1} = O\left(1 + \log \frac{1}{t}\right).$$

**Lemma 4.** If $P_n = 1 + 1/2 + \cdots + 1/(n+1)$, then

(i) \[ \sum_{k=0}^{[n/2]-2} \left| \Delta \left( \frac{P_n(n+1) - P_k(k+1)}{(n-k) \log(n-k+1)} \right) \right| = O(1); \]

(ii) \[ \sum_{k=[n/2]}^{n-2} \left| \Delta \left( \frac{P_k \cdot \frac{1}{k+1} \log(n-k+1)}{\log(n-k+1)} \right) \right| = O\left(\frac{P_n}{n}\right). \]

For proving (i) we observe that

\[ \sum_{k=0}^{[n/2]-2} \left| \Delta \left( \frac{P_n(n+1) - P_k(k+1)}{(n-k) \log(n-k+1)} \right) \right| \]
\[ \leq \sum_{k=0}^{[n/2]-2} \frac{P_n(n+1) - P_k(k+1)}{(n-k)(n-k-1) \log(n-k+1)} \]
\[ + \sum_{k=0}^{[n/2]-2} \frac{P_{k+1}(k+2) - P_k(k+1)}{(n-k) \log(n-k+1)} \]
\[ + O\left( \sum_{k=0}^{[n/2]-2} \frac{P_n(n+1) - P_k(k+1)}{(n-k)^2 \log^2(n-k+1)} \right) \]
\[ = O\left( \sum_{k=0}^{[n/2]-2} \frac{1}{n-k} \right) + O\left[ \sum_{k=0}^{[n/2]-2} \frac{P_{k+1}}{(n-k) \log(n-k+1)} \right] \]
\[ + O\left[ \sum_{k=0}^{[n/2]-2} \frac{1}{(n-k) \log(n-k+1)} \right] \]
\[ = O\left( \sum_{k=0}^{[n/2]-2} \frac{1}{n-k} \right) = O(1). \]
Again
\[
\sum_{k=[n/2]}^{n-2} \left| \Delta \left( \frac{P_k}{k+1} \cdot \frac{1}{\log(n - k + 1)} \right) \right|
\]
\[
= O \left( \sum_{k=[n/2]}^{n-2} \frac{P_k}{k+1} \cdot \frac{1}{(n - k) \log^2(n - k + 1)} \right)
\]
\[
+ O \left( \sum_{k=[n/2]}^{n-2} \frac{P_k}{(k+1)^2 \log(n - k + 1)} \right)
\]
\[
= O\left( \frac{P_n}{n} \right) \left( \sum_{k=[n/2]}^{n-2} \frac{1}{n - k} \log^2(n - k + 1) \right) + O\left( \frac{P_n}{n^2} \right).
\]
This proves the lemma completely.

3. Proof of the theorem. Since
\[
t_n = \frac{P_n u_0 + P_{n-1} u_1 + \cdots + P_0 u_n}{P_n}, \quad \left( u_n = \frac{A_n(t)}{\log(n + 1)} \right),
\]
we have
\[
t_n - t_{n-1} = \sum_{\nu=0}^{n-1} \left( \frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}} \right) u_{n-\nu}
\]
\[
= \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \left( \frac{P_\nu}{\nu + 1} - \frac{P_\nu}{n + 1} \right) u_{n-\nu}.
\]
For the Fourier series of \( f(t) \) at \( t = k \),
\[
A_n = \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt \, dt
\]
so that
\[
t_n - t_{n-1} = \frac{2}{\pi} \int_0^\pi \phi(t) \left( \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\cos(n - k)t}{\log(n - k + 1)} \right) dt.
\]
Thus in order to prove the theorem, we have to establish that
\[
\sum_n \left| \int_0^\pi \phi(t) g(n, t) \, dt \right| < \infty,
\]
where
\[
g(n, t) = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\cos(n - k)t}{\log(n - k + 1)}.
\]
We observe that
\[
\int_0^\pi \phi(t)u(n, t)dt = -\int_0^\pi \left( \int_0^t g(n, u)du \right) \phi(t),
\]
and
\[
\sum_n \left| \int_0^t \left( \int_0^t g(n, u)du \right) \phi(t) \right| \leq \int_0^\pi |\phi(t)| \left\{ \sum_n \left| \int_0^t g(n, u)du \right| \right\}.
\]
Since, by hypothesis, \( \int_0^\pi |\phi(t)| < \infty \), it suffices for our purpose to show that, uniformly for \( 0 < t < \pi \),
\[
\sum_n \left| \int_0^t g(n, u)du \right| < \infty,
\]
or, what is the same thing,
\[
\sum = \sum_n \left| \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k + 1} - \frac{P_k}{n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right| < \infty.
\]
Denoting \( \tau = [1/t] \), we have
\[
\sum \leq \sum_{1}^{\tau} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left( \frac{P_n}{k + 1} - \frac{P_k}{n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|
\]
\[
+ \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \left( \frac{P_n}{k + 1} - \frac{P_k}{n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|
\]
\[
+ \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]} \left( \frac{P_n}{k + 1} - \frac{P_k}{n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|.
\]
Now since \( |\sin(n - k)t| \leq (n - k)t \) and \( P_n(n+1) \geq P_k(k+1) \) for \( k \leq n \), we have
\[
\sum = \sum_{1}^{\tau} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left( \frac{P_n}{k + 1} - \frac{P_k}{n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|
\]
\[
\leq At \sum_{1}^{\tau} \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} P_n/k + 1
\]
\[
= At \sum_{1}^{\tau} P_n P_{n-1}/P_n P_{n-1}
\]
\[
= 0(1).
\]
By Abel's transformation and taking \( r \) to be a fixed number > \( \pi \),
\[\sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \left( \frac{P_n - P_k}{k + 1 \cdot n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right| \]

\[= \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \sin(n - k)t \right| \]

\[= O \left( \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left( \log \frac{r}{t} \right) \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \right) \]

by using Lemmas 3 and 4 (i).

Since \(1/(n+1)(n-k)=1/(k+1)(n-k)-1/(n+1)(k+1)\), we have

\[\sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \left( \frac{P_n - P_k}{k + 1 \cdot n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right| \]

Now since for \(k \geq \lfloor n/2 \rfloor\), \(P_n - P_k = O(1)\), we obtain

\[\sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=\lfloor n/2 \rfloor}^{n-1} \left( \frac{P_n - P_k}{k + 1 \cdot n + 1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right| \]

\[= O \left( \sum_{r+1}^{\infty} \frac{1}{n P_n P_{n-1}} \left| \sum_{k=\lfloor n/2 \rfloor}^{n-1} \frac{1}{k + 1 \cdot n + 1 \cdot (n - k) \log(n - k + 1)} \right| \right) \]

\[= O \left( \sum_{r+1}^{\infty} \frac{\log \log n}{n \log^2 n} \right) = O(1). \]
Finally since $\sum_{a}^{b} \sin nt = O(1/t)$ for all $a$ and $b$, we have by Abel’s transformation

$$\sum_{\delta_{2}} = \sum_{r+1}^{\infty} \frac{1/(n+1)}{P_{n}P_{n-1}} \sum_{k=[n/2]}^{n-1} \frac{P_{k}}{k+1} \frac{\sin(n-k)t}{\log(n-k+1)}$$

$$= O\left(\sum_{r+1}^{\infty} \frac{1/(n+1)}{P_{n}P_{n-1}} \sum_{k=[n/2]}^{n-2} \frac{1}{t} \left| \Delta \left( \frac{P_{k}}{k+1} \frac{1}{\log(n-k+1)} \right) \right| \right)$$

$$+ O\left(\frac{1}{t}\right) \left( \sum_{r+1}^{\infty} \frac{1/(n+1)}{P_{n}P_{n-1}} \frac{P_{n-1}}{n} \right)$$

(3.4)

$$= O\left(\frac{1}{t}\right) \left( \sum_{r+1}^{\infty} \frac{1}{n^{2} \log n} \right)$$

$$= O\left(\frac{1}{t}\right) \left( \frac{1}{r \log r} \right)$$

$$= O(1),$$

using Lemma 4 (ii).

Collecting (3.1), (3.2), (3.3), and (3.4), we see that the theorem is completely proved.

I am very much indebted to Professor Misra for his kind interest and advice in the preparation of this paper.

REFERENCES


University of Saugar, Saugar, India