ON THE ABSOLUTE HARMONIC SUMMABILITY OF A SERIES RELATED TO A FOURIER SERIES

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1.1. Let \( \sum a_n \) be a given infinite series with the sequence of partial sums \( \{s_n\} \). Let the sequence \( \{t_n\} \) be defined by

\[
t_n = \frac{(n+1)^{-1}s_0 + n^{-1}s_1 + \cdots + 1 \cdot s_n}{P_n},
\]

where

\[
(P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}).
\]

The series \( \sum a_n \) is defined to be summable by harmonic means if the sequence \( \{t_n\} \) tends to a limit as \( n \to \infty \) [4]. If the series \( \sum |t_n - t_{n-1}| \) is convergent, we say that the series is absolutely harmonic summable. It is known that the method of summability is absolutely regular and implies absolute Cesàro summability of every positive order [2].

1.2. Let \( f(t) \) be a periodic function, with period \( 2\pi \), and integrable \( (L) \) over \( (-\pi, \pi) \). We assume without any loss of generality that the Fourier series of \( f(t) \) is given by

\[
\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),
\]

and that \( \int_{-\pi}^{\pi} f(t) dt = 0 \). We write

\[
\phi(t) = \frac{1}{2} \{f(x + t) + f(x - t)\}.
\]

Mohanty [3] has considered the absolute Riesz summability of the series

\[
\sum_{n=1}^{\infty} A_n(t)/\log (n + 1).
\]

2. In this paper we establish the following theorem:

**Theorem.** If \( \phi(t) \) is of bounded variation in \( (0, \pi) \) then the series (1.22) is absolutely summable by harmonic means.

We require the following lemmas for the proof of our theorem:

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Lemma 1 [5]. Uniformly for $0 < t < \pi$

$$\left| \sum_{m}^{n} \frac{\sin \nu t}{\nu} \right| \leq K$$

where $m$ and $n$ are any positive integers.

Lemma 2 [1]. If $0 < t < \pi$, then

$$\left| \sum_{k=0}^{m} \cos \left(\frac{(k + 1)t}{k + 1}\right) \right| = O\left(1 + \log \frac{1}{t}\right).$$

With the help of Lemmas 1 and 2, we may easily deduce

Lemma 3. If $0 < t < \pi$, then for all positive integers $m$ and $m'$

$$\sum_{k=m}^{m'} \frac{\sin(n - k)t}{k + 1} = O\left(1 + \log \frac{1}{t}\right).$$

Lemma 4. If $P_n = 1 + 1/2 + \cdots + l/(n+1)$, then

(i) \[ \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \left| \Delta \left( \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \right) \right| = O(1); \]

(ii) \[ \sum_{k=\lfloor n/2 \rfloor}^{n - 2} \left| \Delta \left( \frac{P_k}{k + 1} \frac{1}{\log(n - k + 1)} \right) \right| = O\left(\frac{P_n}{n}\right). \]

For proving (i) we observe that

\[ \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \left| \Delta \left( \frac{P_n(n + 1) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \right) \right| \]

\[ \leq \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \frac{P_n(n + 1) - P_k(k + 1)}{(n - k)(n - k - 1) \log(n - k + 1)} \]

\[ + \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \frac{P_{k+1}(k + 2) - P_k(k + 1)}{(n - k) \log(n - k + 1)} \]

\[ + O\left( \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \frac{P_n(n + 1) - P_k(k + 1)}{(n - k)^2 \log^2(n - k + 1)} \right) \]

\[ = O\left( \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \frac{1}{n - k} \right) + O\left[ \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \frac{P_{k+1}}{(n - k) \log(n - k + 1)} \right] \]

\[ + O\left[ \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \frac{1}{(n - k) \log(n - k + 1)} \right] \]

\[ = O\left( \sum_{k=0}^{\lfloor n/2 \rfloor - 2} \frac{1}{n - k} \right) = O(1). \]
Again
\[
\sum_{k=[n/2]}^{n-2} \left| \Delta \left( \frac{P_k}{k + 1} \cdot \frac{1}{\log(n - k + 1)} \right) \right|
= O \left( \sum_{k=[n/2]}^{n-2} \frac{P_k}{k + 1} \cdot \frac{1}{(n - k) \log^2(n - k + 1)} \right)
+ O \left( \sum_{k=[n/2]}^{n-2} \frac{P_k}{(k + 1)^2 \log(n - k + 1)} \right)
= O(P_n/n) \left( \sum_{k=[n/2]}^{n-2} \frac{1}{(n - k) \log^2(n - k + 1)} \right) + O\left( \frac{P_n}{n^2} \right) \cdot (n)
= O(P_n/n).
\]
This proves the lemma completely.

3. Proof of the theorem. Since
\[
l_n = \frac{P_n u_0 + P_{n-1} u_1 + \cdots + P_0 u_n}{P_n}, \quad \left( u_n = \frac{A_n(t)}{\log(n + 1)} \right),
\]
we have
\[
l_n - l_{n-1} = \sum_{r=0}^{n-1} \left( \frac{P_r}{P_n} - \frac{P_{r-1}}{P_{n-1}} \right) u_{n-r}
= \frac{1}{P_n P_{n-1}} \sum_{r=0}^{n-1} \left( \frac{P_n}{r + 1} - \frac{P_r}{n + 1} \right) u_{n-r}.
\]
For the Fourier series of \( f(t) \) at \( t = k \),
\[
A_n = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos(nt) \, dt
\]
so that
\[
l_n - l_{n-1} = \frac{2}{\pi} \int_0^{\pi} \phi(t) \left( \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k + 1} - \frac{P_k}{n + 1} \right) \frac{\cos((n - k)t)}{\log(n - k + 1)} \right) \, dt.
\]
Thus in order to prove the theorem, we have to establish that
\[
\sum_n \left| \int_0^{\pi} \phi(t) g(n, t) \, dt \right| < \infty,
\]
where
\[
g(n, t) = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k + 1} - \frac{P_k}{n + 1} \right) \frac{\cos((n - k)t)}{\log(n - k + 1)}.
\]
We observe that
\[
\int_0^\tau \phi(t) g(n, t) dt = - \int_0^\tau \left( \int_0^t g(n, u) du \right) d\phi(t),
\]
and
\[
\sum_n \left| \int_0^\tau \left( \int_0^t g(n, u) du \right) d\phi(t) \right| \leq \int_0^\tau \left| d\phi(t) \right| \left\{ \sum_n \left| \int_0^t g(n, u) du \right| \right\}.
\]
Since, by hypothesis, \( \int_0^\tau \left| d\phi(t) \right| < \infty \), it suffices for our purpose to show that, uniformly for \( 0 < t < \pi \),
\[
\sum_n \left| \int_0^t g(n, u) du \right| < \infty,
\]
or, what is the same thing,
\[
\sum = \sum_n \left| \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right| < \infty.
\]
Denoting \( \tau = [1/t] \), we have
\[
\sum \leq \sum_{1}^{r} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|
\]
\[
+ \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|
\]
\[
+ \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{[n/2]}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|.
\]
Now since \( |\sin(n - k)t| \leq (n - k)t \) and \( P_n(n+1) \geq P_k(k+1) \) for \( k \leq n \), we have
\[
\sum = \sum_{1}^{r} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n - k)t}{(n - k) \log(n - k + 1)} \right|
\]
\[
\leq At \sum_{1}^{r} \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} P_n/k + 1\]
\[
= At \sum_{1}^{r} \frac{P_n P_{n-1}}{P_n P_{n-1}}
\]
\[
= o(1).
\]
By Abel’s transformation and taking \( r \) to be a fixed number \( > \pi \),
\[
\begin{align*}
\sum_{2}^{\infty} &= \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k) \log(n-k+1)} \right| \\
&= \sum_{r+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left| \sum_{k=0}^{[n/2]-1} \frac{P_n(n+1) - P_k(k+1)}{(n-k) \log(n-k+1)} \frac{\sin(n-k)t}{k+1} \right| \\
&= O \left[ \sum_{r+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left( \log \frac{r}{l} \right)^{[n/2]-2} \sum_{k=0}^{[n/2]-1} \left| \Delta \frac{P_n(n+1) - P_k(k+1)}{(n-k) \log(n-k+1)} \right| \right] \\
&= O \left( \log \frac{r}{l} \right) \sum_{r+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} = O \left( \frac{\log r/l}{P_r} \right) \\
&= O(1),
\end{align*}
\]

by using Lemmas 3 and 4 (i).

Since \(1/(n+1)(n-k) = 1/(k+1)(n-k) - 1/(n+1)(k+1)\), we have

\[
\sum_{3}^{\infty} = \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \left( \frac{P_n}{k+1} - \frac{P_k}{n+1} \right) \frac{\sin(n-k)t}{(n-k) \log(n-k+1)} \right| \\
\leq \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_n - P_k}{(k+1) \log(n-k+1)} \frac{\sin(n-k)t}{k+1} \right| \\
+ \sum_{r+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_k}{(n-k) \log(n-k+1)} \frac{\sin(n-k)t}{k+1} \right| \\
= \sum_{31}^{\infty} + \sum_{32}^{\infty},
\]

say.

Now since for \(k \geq [n/2]\), \(P_n - P_k = O(1)\), we obtain

\[
\sum_{31}^{\infty} = \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_n - P_k}{k+1} \frac{\sin(n-k)t}{(n-k) \log(n-k+1)} \right| \\
= O \left[ \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_n - P_k}{(k+1) \log(n-k+1)} \right| \right] \\
= O \left( \log \log \frac{n}{n \log^2 n} \right) = O(1).
\]
Finally since $\sum_{a}^{b} \sin nt = O(1/t)$ for all $a$ and $b$, we have by Abel’s transformation

$$\sum_{32}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \left| \sum_{k=[n/2]}^{n-1} \frac{P_k}{k+1} \frac{\sin(n-k)t}{\log(n-k+1)} \right|$$

$$= O\left( \sum_{r+1}^{\infty} \frac{1/(n+1)}{P_n P_{n-1}} \frac{1}{t} \Delta \left( \frac{P_k}{k+1} \log(n-k+1) \right) \right)$$

$$+ O\left( \frac{1}{t} \left( \sum_{r+1}^{\infty} \frac{1}{P_n P_{n-1}} \frac{1}{n^2 \log n} \right) \right)$$

$$= O\left( \frac{1}{t} \left( \sum_{r+1}^{\infty} \frac{1}{n^2 \log n} \right) \right)$$

$$= O\left( \frac{1}{t} \left( \frac{1}{r \log r} \right) \right)$$

$$= O(1),$$

using Lemma 4 (ii).

Collecting (3.1), (3.2), (3.3), and (3.4), we see that the theorem is completely proved.

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References


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