CERTAIN COLLECTIONS OF ARCS IN $E^3$

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1. Introduction. In considering upper semicontinuous decompositions of $E^3$, it is sometimes useful to know whether a given collection of continua can be transformed, by a homeomorphism of $E^3$ onto itself, into another collection which is simpler in some respects; for example, a collection of straight line intervals might be transformed into a collection of vertical intervals, or a collection of arcs into a collection of straight line intervals. It might also be useful to know conditions under which such a transformation can be effected by means of a particular type of homeomorphism of $E^3$ onto itself.

In this paper, the following questions of this type will be considered. Suppose $\alpha$ and $\beta$ are horizontal planes and $G$ is a continuous collection of mutually exclusive arcs, each of which is irreducible from $\alpha$ to $\beta$ and no one of which contains two points of any horizontal plane, such that the sum of the elements of $G$ is compact and intersects $\alpha$ in a totally disconnected set. Under what conditions is there a homeomorphism of $E^3$ onto itself which takes each element of $G$ onto a vertical interval and does not change the $z$-coordinate of any point?

It is shown, with the aid of certain results due to Bing [1] and Fort [5], that such a transformation is not always possible, even when the elements of $G$ are straight line intervals. The following condition is found to be necessary and sufficient for the existence of such a transformation (see §3 for definitions of unfamiliar terms): For every positive number $\epsilon$ there exists a finite set $K_1, K_2, \ldots, K_n$ of topological cylinders with bases on $\alpha$ and $\beta$ such that (1) the solid cylinders determined by $K_1, K_2, \ldots, K_n$ are mutually exclusive, (2) each arc of $G$ is enclosed by some $K_i$ and (3) each $K_i$ has horizontal diameter less than $\epsilon$.

2. Examples. The decomposition given by Bing in [1] can be modified so that the collection of nondegenerate elements is of the type considered above. If there were a homeomorphism of $E^3$ onto itself carrying these arcs onto vertical intervals, then by [2, Theorem 5], the decomposition space would be homeomorphic to $E^3$; since this is not the case, there is no such homeomorphism.

A stronger example is furnished by Fort's modification [5] of Bing's example. This modification can be carried out in such a way

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that there exist four horizontal planes $\alpha$, $\beta$, $\gamma$, $\delta$ with $\alpha$ above $\beta$, $\beta$ above $\gamma$, and $\gamma$ above $\delta$, such that each nondegenerate element of the decomposition is the sum of three intervals $g_1$, $g_2$, $g_3$ with end points on $\alpha$ and $\beta$, $\beta$ and $\gamma$, and $\gamma$ and $\delta$, respectively. Let $G$ denote the collection of all nondegenerate elements of this decomposition and let $G_1$, $G_2$ and $G_3$ denote, respectively, the collection of all intervals lying in an element of $G$ and having end points on $\alpha$ and $\beta$, the collection of all such intervals having end points on $\beta$ and $\gamma$, and the collection of all those with end points on $\gamma$ and $\delta$. Suppose there is a homeomorphism of $E^3$ onto itself which does not change the $z$-coordinate of any point and which takes each element of $G_1$ onto a vertical interval. Then there is a homeomorphism $f_1$ of $E^3$ onto itself which is fixed on $\beta$ and on all points below $\beta$ and which takes each element of $G_1$ onto a vertical interval. From the symmetry of the construction of $G$, it follows that there is also a homeomorphism $f_2$ of $E^3$ onto itself which is fixed on $\gamma$ and all points above $\gamma$ and which takes each element of $G_3$ onto a vertical interval. Then $f_2f_1$ is a homeomorphism of $E^3$ onto itself which is fixed on $\beta$, $\gamma$ and all points between $\beta$ and $\gamma$ and which takes each element of $G_1 + G_3$ onto a vertical interval. The proof of the main theorem below shows that there is a homeomorphism $f_2$ of $E^3$ onto itself which is fixed on $\gamma$ and all points below $\gamma$, does not change the $z$-coordinate of any point, takes each element of $G_2$ onto a vertical interval, and is such that if $p$ and $q$ are two points above $\beta$ and on the same vertical line, then $f_2(p)$ and $f_2(q)$ are on the same vertical line. The transformation $f_2f_1$ is a homeomorphism of $E^3$ onto itself which takes each element of $G$ onto a vertical interval. But this is impossible since it implies, as before, that the decomposition space is homeomorphic to $E^3$. Hence the elements of $G_1$ cannot be transformed into a collection of vertical intervals by a homeomorphism of $E^3$ onto itself which does not change the $z$-coordinate of any point.

It is perhaps worth noting that if $G_1'$ is the decomposition of $E^3$ whose only nondegenerate elements are the elements of $G_1$, then the decomposition space of $G_1'$ is homeomorphic to $E^3$. This is a consequence of the following theorem, which is essentially proved in [2].

If $G$ is a monotone upper semicontinuous decomposition of $E^3$ such that (1) the set of nondegenerate elements of $G$ is 0-dimensional in the decomposition space and (2) for every positive number $\epsilon$ and every open set $U$ containing the sum of the nondegenerate elements of $G$, there is a homeomorphism of $E^3$ onto itself which is fixed on $E^3 - U$ and which takes each element of $G$ into a set of diameter less than $\epsilon$, then the decomposition space is homeomorphic to $E^3$. 
3. Definitions. If \( G \) is a collection of sets, then \( G^\ast \) will denote the sum of the elements of \( G \); the collection \( G \) is said to fill up a point set \( M \) if \( G^\ast = M \).

A subset \( K \) of \( E^3 \) will be called a topological cylinder provided there exist two parallel planes \( \alpha_0 \) and \( \alpha_1 \) and a continuous collection \( G \) of mutually exclusive arcs filling up \( K \) such that (1) each element of \( G \) is irreducible from \( \alpha_0 \) to \( \alpha_1 \), (2) no element of \( G \) contains two points of any plane parallel to \( \alpha_0 \), and (3) \( \alpha_0 \cdot K \) and \( \alpha_1 \cdot K \) are simple closed curves. The planar disks bounded by \( \alpha_0 \cdot K \) and \( \alpha_1 \cdot K \) will be called the bases of \( K \) and the collection \( G \) will be called a set of generators for \( K \). A topological cylinder plus its bases will be called a closed topological cylinder and a closed topological cylinder plus its interior will be called a solid topological cylinder.

If \( A \) is a topological cylinder with bases on the planes \( \alpha_0 \) and \( \alpha_1 \), then \( K \) is said to enclose a point set \( M \) provided that (1) each point of \( M \) lies either between the planes \( \alpha_0 \) and \( \alpha_1 \) or else on one of those planes and (2) if \( \alpha \) is a plane parallel to \( \alpha_0 \) or \( \alpha_1 \) and intersecting \( M \), then the simple closed curve \( \alpha \cdot K \) encloses \( \alpha \cdot M \) (i.e., \( \alpha \cdot M \) is a subset of the bounded component of \( \alpha - \alpha \cdot K \)).

If \( K \) is a topological cylinder with horizontal bases, then \( \max(\text{dia}(\alpha \cdot K)) \), \( \alpha \) a horizontal plane, will be called the horizontal diameter of \( K \).

4. Lemma 1. Suppose \( A \) is the annulus bounded by the unit circle \( C_1 \) and the circle \( C_2 \) with center \( O \) and radius \( 2 \), and \( G \) is a collection of mutually exclusive arcs filling up \( A \) such that each arc of \( G \) has one end point on \( C_1 \) and the other on \( C_2 \) and no arc of \( G \) contains two points of any circle with center \( O \). Then there exists an isotopy \( \{ F_t \} \), \( 0 \leq t \leq 1 \), such that (1) for each \( t \), \( F_t \) is a homeomorphism of \( A \) onto itself which does not change the distance from \( O \) of any point, (2) \( F_0 \) is the identity on \( A \) and (3) \( F_t \) is a homeomorphism which takes each element of \( G \) into an interval lying on a line through \( O \).

Proof. Let \( g_0 \) be an element of \( G \). There is a continuous function \( \phi(r) \), \( 1 \leq r \leq 2 \), such that \( g_0 \) has the polar coordinate equation \( \theta = \phi(r) \). For each \( t \) in \([0, 1/2]\) and each point \((r, \theta)\) of \( A \), let \( F^1_t(r, \theta) = (r, \theta - 2t \cdot \phi(r)) \). Then \( \{ F^1_t \} \), \( 0 \leq t \leq 1/2 \), is an isotopy on \( A \) and \( F^1_0 \) is the identity. If \((r, \theta) \in g_0 \), then \( F^1_{1/2}(r, \theta) = (r, 0) \), so \( F^1_{1/2} \) takes \( g_0 \) onto the interval with end points \((1, 0)\) and \((2, 0)\).

Let \( g'_0 = F^1_{1/2}(g_0) \) and let \( G' \) denote the collection of all images under \( F^1_{1/2} \) of elements of \( G \). For each point \( p \) of \( A \), let \( \theta(p) \) be the smallest non-negative polar angle for \( p \) and let \( \pi(p) \) denote the point of intersection of \( C_1 \) and the arc of \( G' \) containing \( p \). For each \( t \) in \([1/2, 1]\)
and each point \( p = (r, \theta) \) of \( A \), let \( F_t^2(p) = (r, 2(1-t)\cdot \theta(p) + (2t-1)\cdot \theta(\pi(p))) \).

Since \( \pi(p) \) is continuous on \( A \) and \( \theta(p) \) is continuous on \( A - g'_0 \), \( F_t^2 \) is continuous on \( A - g'_0 \). By a direct argument, it can be shown that \( F_t^2 \) is also continuous at each point of \( g'_0 \), so it is continuous on all of \( A \). From the fact that if \( \theta(p_1) < \theta(p_2) \), then \( \theta(\pi(p_1)) < \theta(\pi(p_2)) \), it follows that \( F_t^2 \) is 1-1 and hence is a homeomorphism. It is easily verified that \( \{F_t^2\}, 1/2 \leq t \leq 1 \), is an isotopy. Clearly \( F_{1/2}^2 \) is the identity on \( A \), and since for each \( p \) in \( A \), \( \theta(\pi(p)) \) is a polar angle for \( F_t^2(p) \) and \( \pi(p) \) is constant on any element of \( G' \), \( F_t^2 \) takes each element of \( G' \) into an interval lying on a line through \( O \).

For each \( t \) in \([0, 1/2]\), let \( F_t = F_t^1 \) and for each \( t \) in \([1/2, 1]\), let \( F_t = F_t^2F_{1/2}^1 \). Then \( \{F_t\}, 0 \leq t \leq 1 \), is an isotopy satisfying the desired conditions.

**Lemma 2.** Suppose \( K_1 \) and \( K_2 \) are right circular cylinders with horizontal bases such that \( K_2 \) encloses \( K_1 \). For \( * = 1, 2 \), let \( G_* \) be a set of generators for \( K_* \) and let \( U_* \) denote the interior of the closed cylinder determined by \( K_* \). Then there exists a continuous collection \( G \) of mutually exclusive arcs filling up the closure of \( U_2 - U_1 \) such that no element of \( G \) contains two points of any horizontal plane and such that \( G_1 + G_2 \subseteq G \).

**Proof.** Suppose \( K_i, i = 1, 2 \), is represented in cylindrical coordinates by the equations \( r = i, 0 \leq z \leq 1 \). It follows from Lemma 1 that there is an isotopy \( \{F_t^2\}, 1 \leq t \leq 3/2 \), such that (1) for each \( t \) in \([1, 3/2]\), \( F_t^1 \) is a homeomorphism of \( K_1 \) onto itself which does not change the \( z \)-coordinate of any point, (2) \( F_{3/2} \) is the identity on \( K_1 \) and (3) \( F_t^1 \) takes each element of \( G_1 \) onto a vertical interval. Similarly, there exists an isotopy \( \{F_t^2\}, 3/2 \leq t \leq 2 \), such that for each \( t \) in \([3/2, 2]\), \( F_t^2 \) is a homeomorphism of \( K_2 \) onto itself which does not change the \( z \)-coordinate of any point, (2) \( F_{3/2}^2 \) is the identity on \( K_2 \) and (3) \( F_t^2 \) takes each element of \( G_2 \) onto a vertical interval.

Let \( M = \text{Cl}(U_2 - U_1) \) and for each point \( p = (r, \theta, z) \) of \( M \), let \( F(p) \) be the point \( (r, \theta', z) \), where \( \theta' \) is such that if \( r \leq 3/2 \), \( F_t^1(1, \theta, z) = (1, \theta', z) \) and if \( r \geq 3/2 \), \( F_t^2(2, \theta, z) = (2, \theta', z) \). Then \( F \) is a homeomorphism of \( M \) onto itself which does not change the \( z \)-coordinate of any point. Since \( F \) agrees with \( F_t^1 \) on \( K_1 \) and with \( F_t^2 \) on \( K_2 \), it takes each element of \( G_1 + G_2 \) onto a vertical interval.

Let \( G' \) denote the collection of all vertical intervals lying in \( M \) and having one end point on \( \alpha_0 \) and the other on \( \alpha_1 \) and let \( G \) denote the collection of all images under \( F^{-1} \) of elements of \( G' \). Then \( G \) is a collection of mutually exclusive arcs filling up \( M \) and satisfying the desired conditions.
Lemma 3. Suppose $K_0, K_1, K_2, \ldots, K_n$ are topological cylinders with bases on the horizontal planes $\alpha_0$ and $\alpha_1$ such that $K_0$ encloses $K_j$ ($j=1, 2, \ldots, n$) and such that no two of the solid cylinders determined by $K_1, K_2, \ldots, K_n$ have a point in common. If for $j=0, 1, 2, \ldots, n$, $G_j$ is a set of generators for $K_j$ and $U_j$ is the interior of the closed cylinder determined by $K_j$, then there is a continuous collection $G$ of mutually exclusive arcs filling up the closure of $U_0 - (U_1 + U_2 + \cdots + U_n)$ such that (1) no element of $G$ contains two points of any horizontal plane and (2) each of $G_0, G_1, \ldots, G_n$ is a subcollection of $G$.

Proof. Let $K'_0, \ldots, K'_n$ denote right circular cylinders with bases on $\alpha_0$ and $\alpha_1$ which are related in the same way as the correspondingly lettered topological cylinders $K_0, \ldots, K_n$. Let $U'_j$, $j=0, 1, \ldots, n$, denote the interior of $K'_j$ and let $M$ and $M'$ denote, respectively, the closures of $U_0 - (U_1 + \cdots + U_n)$ and $U'_0 - (U'_1 + \cdots + U'_n)$. It follows from Theorem 1 and Lemma 2 of [4] and the remark following the proof of Theorem 5 of [3] that there is a homeomorphism $h$ of $M$ onto $M'$ which does not change the $z$-coordinate of any point. For $j=0, 1, \ldots, n$, let $G'_j$ denote the collection of all images under $h$ of elements of $G_j$.

Let $C_0$ be a right circular cylinder with bases on $\alpha_0$ and $\alpha_1$ which is enclosed by $K'_0$ and encloses each of $K'_1, \ldots, K'_n$. Let $C_1, \ldots, C_n$ be right circular cylinders which determine mutually exclusive solid cylinders, such that $C_j$ encloses $K'_j$ and is enclosed by $C_0$. Let $V_j$ denote the interior of $C_j$, let $M_0 = \text{Cl}(U'_0 - V_0)$ and for $j=1, 2, \ldots, n$, let $M_j = \text{Cl}(V_j - U'_j)$. It follows from Lemma 2 that, for $j=0, 1, \ldots, n$, there exists a continuous collection $H_j$ of mutually exclusive arcs filling up $M_j$ such that no element of $H_j$ contains two points of any horizontal plane, $G'_j \subset H_j$, and every element of $H_j$ which intersects $C_j$ is a vertical interval. Let $H = H_1 + H_2 + \cdots + H_n$ and let $G'$ denote the collection obtained by adding to $H$ all vertical intervals with end points on $\alpha_0$ and $\alpha_1$ which intersect $M'$. Then the collection $G$ of all images under $h^{-1}$ of elements of $G'$ satisfies the desired conditions.

Theorem. Suppose $\alpha_0$ and $\alpha_1$ are horizontal planes and $G$ is a continuous collection of mutually exclusive arcs such that (1) each element of $G$ is irreducible from $\alpha_0$ to $\alpha_1$ and no element of $G$ contains two points of any horizontal plane, and (2) $G^*$ is compact and intersects $\alpha_0$ in a totally disconnected set. In order that there should exist a homeomorphism of $E^3$ onto itself which takes each element of $G$ onto a vertical interval and
does not change the \( z \)-coordinate of any point, it is necessary and sufficient that for every positive number \( \epsilon \), there exist a finite set \( K_1, K_2, \ldots, K_n \) of topological cylinders with bases on \( \alpha_0 \) and \( \alpha_1 \) such that (1) the solid cylinders determined by \( K_1, K_2, \ldots, K_n \) are mutually exclusive, (2) each arc of \( G \) is enclosed by some \( K_i \) and (3) each \( K_i \) has horizontal diameter less than \( \epsilon \).

PROOF. 1. Suppose there is a homeomorphism \( h \) of \( E^3 \) onto itself which takes each element of \( G \) onto a vertical interval and does not change the \( z \)-coordinate of any point. Let \( G' \) denote the set of images under \( h \) of the elements of \( G \) and let \( K' \) be a vertical cylinder with bases on \( \alpha_0 \) and \( \alpha_1 \) which encloses \( G'^* \).

Suppose \( \epsilon \) is a positive number. Let \( S \) be a compact set containing the solid cylinder determined by \( K' \) in its interior. There is a positive number \( \delta \) such that if \( p \) and \( q \) are points of \( S \) and \( \rho(p, q) < \delta \), then \( \rho(h^{-1}(p), h^{-1}(q)) < \epsilon \). Since \( \alpha_0 \cdot G'^* \) is compact and totally disconnected, there exists a finite set \( D_1, D_2, \ldots, D_n \) of mutually exclusive disks in \( \alpha_0 \), each of diameter less than \( \epsilon \), such that every point of \( \alpha_0 \cdot G'^* \) is in the interior of some \( D_i \). Let \( K'_i, i = 1, 2, \ldots, n \), denote the topological cylinder having \( D_i \) as one of its bases and having its other base on \( \alpha_1 \), which has a collection of vertical intervals as a set of generators. If \( K_i = h^{-1}(K'_i) \), then \( K_1, K_2, \ldots, K_n \) satisfy the conditions of the theorem.

2. Suppose the condition is satisfied. Let \( K \) be a topological cylinder having a set of vertical generators, such that the bases of \( K \) are on \( \alpha_0 \) and \( \alpha_1 \) and \( K \) encloses \( G^* \). By hypothesis, there exists a sequence \( H_1, H_2, H_3, \ldots \) such that (1) for each \( n \), \( H_n \) is a finite collection of cylinders each having one base on \( \alpha_0 \) and the other on \( \alpha_1 \), such that no two of the solid cylinders determined by the elements of \( H_n \) have a point in common, (2) \( K \) encloses each element of \( H_i \) and for each \( n \), each element of \( H_n+1 \) is enclosed by some element of \( H_n \), (3) for each \( n \), each arc of \( G \) is enclosed by some element of \( H_n \), and (4) for each \( n \), each element of \( H_n \) has horizontal diameter less than \( 1/n \).

Let \( U \) denote the interior of the closed cylinder determined by \( K \) and for each \( n \), let \( U_n \) denote the sum of the interiors of the closed cylinders determined by the elements of \( H_n \).

Let \( G_0 \) be the set of vertical generators for \( K \). It follows from Lemma 3 that there exists a continuous collection \( G_1 \) of mutually exclusive arcs filling up the closure of \( U - U_i \) such that (1) each element of \( G_1 \) is irreducible from \( \alpha_0 \) to \( \alpha_1 \) and no element of \( G_1 \) contains two points of any horizontal plane and (2) \( G_0 \subset G_1 \) and each arc of \( G_1 \) which intersects an element of \( H_i \) is a subset of that element. By
applying Lemma 3 to each element of $H_1$, it can be shown that there is a continuous collection $G_2$ of mutually exclusive arcs filling up the closure of $U - U_2$, satisfying the first condition imposed on $G_1$ above and such that $G_1 \subset G_2$ and each arc of $G_2$ which intersects an element of $H_2$ is a subset of that element. By continuing this process, there may be obtained a sequence $G_1, G_2, G_3, \ldots$ such that (1) for each $n$, $G_n$ is a continuous collection of mutually exclusive arcs filling up the closure of $U - U_n$ such that each element of $G_n$ is irreducible from $\alpha_0$ to $\alpha_1$ and no element of $G_n$ contains two points of any horizontal plane, and (2) for each $n$, $G_n \subset G_{n+1}$. Let $G' = G + G_1 + G_2 + \cdots$. Then $G'$ is a continuous collection of mutually exclusive arcs filling up the solid cylinder determined by $K$, each element of $G'$ is irreducible from $\alpha_0$ to $\alpha_1$, no element of $G'$ contains two points of any horizontal plane, and each element of $G'$ which intersects $K$ is a vertical interval.

Let $M$ denote the solid cylinder determined by $K$. For each point $p$ of $M$, let $f(p)$ be that point $q$ on the horizontal plane containing $p$ such that the projection of $q$ onto $\alpha_0$ is an end point of the arc of $G'$ containing $p$. Then $f$ is a homeomorphism of $M$ onto itself which is fixed on $K$ and on $M \cdot \alpha_0$, does not change the $z$-coordinate of any point, and takes each element of $G'$ onto a vertical interval. Let $F$ be the function which agrees with $f$ on $M$, leaves fixed each point of $E^3 - M$ not lying directly above a point of $M$, and is such that if $p$ is a point of $E^3 - M$ lying directly above the point $q$ of $\alpha_1 \cdot M$ (supposing $\alpha_1$ is above $\alpha_0$), then $F(p)$ is the point with the same $z$-coordinate as $p$ which lies directly above the point $f(q)$. Then $F$ is a homeomorphism of $E^3$ onto itself which satisfies all the desired conditions.

References

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