ON INEQUALITIES WITH ALTERNATING SIGNS

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1. Introduction. In a recent paper, Olkin [2], established the following result.

**Theorem 1.** Let

(a) \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, \)

(1)

(b) \( 1 \geq w_1 \geq w_2 \geq \cdots w_m \geq 0, \)

(c) \( h(x) \) be convex in \([0, a_1], h(0) \leq 0. \)

Then

(2) \[ \sum_{j=1}^{m} (-1)^{i-1}w_jh(a_j) \geq h \left( \sum_{j=1}^{m} (-1)^{i-1}w_ja_j \right). \]

This is an extension of the result given in [1], which in turn is an extension of the original result of Weinberger, [4]. The purpose of this paper is to show that Olkin's result is a special case of an interesting inequality due to Steffensen, [3].

2. Steffensen's inequality. The result of Steffensen is the following.

**Theorem 2.** Let

(a) \( f(t) \) be non-negative and monotone decreasing in \([a, b].\)

(1)

(b) \( g(t) \) satisfy the constraint \( 0 \leq g(t) \leq 1, t \) in \([a, b].\)

Then

(2) \[ \int_{b-c}^{b} f(t) \, dt \leq \int_{a}^{b} f(t) g(t) \, dt \leq \int_{a}^{a+c} f(t) \, dt, \]

where

(3) \[ c = \int_{a}^{b} g(t) \, dt. \]

Let us give a proof for the sake of completeness. Define the function \( u(s) \) by the relation

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1 This is an appropriate place to note that the results in [1] and [4] are themselves special cases of Theorem 108 of Inequalities by Hardy, Littlewood and Pólya.

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It is easy to see that $u(a) = a$, that $u(s)$ is continuous and monotone increasing as $s$ goes from $a$ to $b$, and that $u(s) \leq s$. The condition that $0 \leq g(t) \leq 1$ is essential here. We have, upon differentiating,

$$
(5) \quad f(u) \frac{du}{ds} = f(s)g(s),
$$

whence,

$$
(6) \quad \frac{du}{ds} = \frac{f(s)g(s)}{f(u)} \leq g(s),
$$

taking account of the fact that $u(s) \leq s$ and that $f(s)$ is monotone decreasing. Hence,

$$
(7) \quad u \leq a + \int_a^s g(s)\,ds.
$$

This yields the right-hand side of (2), and the left-hand side is derived in the same fashion.

3. **Olkin’s inequality.** To derive Olkin’s result, choose for $g(t)$ the function defined by

$$
(1) \quad g(t) = \lambda_k, \quad a_{k+1} \leq t \leq a_k, \quad k = 1, 2, \ldots, m - 1,
$$

where $\lambda_1 = w_1$, $\lambda_2 = w_1 - w_2$, $\lambda_3 = w_1 - w_2 + w_3$, and so on, and for $h(t)$ the function defined by

$$
(2) \quad h'(t) = f(t).
$$

Using the inequality of (2.2) for the preceding choice of functions, we obtain a slightly stronger result of the form of (1.2).

4. **A generalization of Steffensen’s inequality.** Let us now establish a generalization of the inequality given in §2. It will be clear that many further results of this type can be obtained using the same techniques.

**Theorem 3.** Let

(a) $f(t)$ be non-negative and monotone decreasing in $[a, b]$.

(b) $f \in L^p[a, b]$.

(c) $g(t) \geq 0$ in $[a, b]$ and $\int_a^b g^p \,dt \leq 1$, 

where \( p > 1 \) and \( 1/p + 1/p' = 1 \). Then

\[
\left( \int_a^b f g dt \right)^p \leq \int_a^b f^p dt,
\]

where

\[
c = a + \left( \int_a^b g dt \right)^p.
\]

**Proof.** Consider the function \( u(t) \) defined for \( a \leq t \leq b \) by the equation

\[
\left( \int_a^t f g dt \right)^p = \int_a^t f^p dt.
\]

Since

\[
\left( \int_a^t f g dt \right)^p \leq \left( \int_a^t f^p dt \right) \left( \int_a^t g^p dt \right)^{p/p'} \leq \int_a^t f^p dt,
\]

we see that \( u(t) \) exists and satisfies the relation \( u(t) \leq t \) for \( t \) in \([a, b]\) with \( u(a) = a \). This function is monotone increasing and satisfies the differential equation

\[
f(u)^p \frac{du}{dt} = pf(t)g(t) \left( \int_a^t f g dt \right)^{p-1}
\]

almost everywhere.

The monotonic nature of \( f(t) \) and \( u(t) \) yield the inequality

\[
\frac{du}{dt} \leq pg(t) \left( \int_a^t g dt \right)^{p-1},
\]

whence

\[
u(t) \leq a + \left( \int_a^t g dt \right)^p.
\]

This completes the proof.

**References**


