THE LEBESGUE CONSTANTS FOR JACOBI SERIES, I

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1. **Introduction.** The nth partial sum of the (Fourier) expansion of an arbitrary function in terms of a given set of orthogonal polynomials can be written as an integral [**5**, p. 38] analogous to Dirichlet's familiar representation in the theory of classical Fourier series. By means of the Christoffel-Darboux formula [**5**, p. 42] this analogue is seen to share also the property of being what Lebesgue called a "singular integral" and whose theory he developed in [**1**].[2]

His work showed, i.a., that the unboundedness (in n) of the norm of the singular integral is necessary and sufficient for the existence of a continuous function whose representation in this manner fails to converge to the function at a preassigned point.

This norm, depending on n as well as on the point in question, is known as the nth *Lebesgue constant* (at the preassigned point).

The importance of this sequence has led many investigators to concern themselves not only with the fact of boundedness or unboundedness but also with the asymptotic expansion as a function of n, in case the sequence is unbounded. In each case which has come to this writer's notice, the order of magnitude of the Lebesgue constants has shown a close relationship with the least order of Cesàro summation effective for the development in question, namely, if the nth Lebesgue constant is precisely of order n^k, then the development of a continuous function is summable $(C, k+\epsilon)$ for each $\epsilon>0$, but not necessarily for $\epsilon=0$; for order $\log n$, the (C, ϵ) method is effective under the same restrictions.

This note is concerned with *Jacobi series*, by which is meant here a development in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha>-1$, $\beta>-1$,[3] at the end-point[4] $x=1$ of the interval of orthogonality.

Such series are (C, k)-summable for any $k>\alpha+1/2$, but not necessarily for $k=\alpha+1/2$, for continuous functions [**5**, p. 239, Theorem 9.1.3].

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[2] Cf. also [**5**, pp. 12–14].

[3] This limitation on α and β is retained throughout both parts.

[4] The Lebesgue constants associated with the other end-point $x=-1$, say $L_n(\alpha, \beta; -1)$, coincide with $L_n(\beta, \alpha)$, as is clear from (9).

2. **Statement of results.** The Lebesgue constants, $L_n(\alpha, \beta)$, for Jacobi series were discussed first in H. Rau's dissertation [**3**]. His representation [**3**, p. 249, (40)] becomes, on changing the variable by $x = \cos\theta$,

$$(1)\quad L_n(\alpha, \beta) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \int_0^\pi (\sin\{\theta/2\})^{2\alpha+1}(\cos\{\theta/2\})^{2\beta+1} \cdot \left| P_n^{(\alpha+1,\beta)}(\cos\theta) \right| d\theta.$$

He showed that, for $\alpha > -1/2$,

$$(2)\qquad L_n(\alpha, \beta) = A_{\alpha\beta} n^{\alpha+1/2} + o(n^{\alpha+1/2}), \qquad n \to \infty,^5$$

where

$$(3)\qquad A_{\alpha\beta} = \frac{2}{\pi^{3/2}} \frac{\Gamma(\alpha/2+1/4)\Gamma(\beta/2+3/4)}{\Gamma(\alpha+1)\Gamma([\alpha+\beta]/2+1)},$$

and mentioned two results communicated orally to him by G. Szegö:[6]

$$(4)\qquad L_n(-1/2, \beta) = \frac{4}{\pi^2} \log n + o(\log n),$$

and

$$(5)\qquad L_n(\alpha, \beta) = \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\infty \theta^\alpha \left| J_{\alpha+1}(\theta) \right| d\theta + o(1),$$

for $-1 < \alpha < -1/2$, where $J_{\alpha+1}(\theta)$ is the Bessel function of first kind and $(\alpha+1)$st order. (For this range of α, the integral in (5) converges and so, hence, does the Jacobi series for any continuous function, in conformity with [**5**, p. 239, Theorem 9.1.3].)

In this paper and its sequel these results are sharpened, particularly in the cases $\alpha = -1/2$ and $-1/2 < \alpha < 1/2$. The case $\alpha = -1/2$ includes the case of Tchebycheff polynomials (where also $\beta = -1/2$) which is essentially[7] the instance of classic Fourier series, while

[5] All O- and o-terms are taken as the parameter becomes infinite.

[6] Proofs of (4) and (5), together with an alternative proof of (2), have been published by Szegö [**4**, §20, p. 87].

[7] In [**5**, p. 59] the top lines of (4.1.7) and (4.1.8) give the expressions involved here. The presence of certain numerical factors there which are absent from the corresponding expressions in Fourier series should be noted. The second corresponds, as indicated in (1) here, to the Dirichlet kernel. It will be shown in Part II that the nth Lebesgue constant in the Tchebycheff case has not only the same principal (logarithmic) term, but even the same constant term as the corresponding Lebesgue constant in Fourier series. While not surprising, this fact is not an automatic consequence of the similarity between the two series (cf. [**2**, last paragraph, p. 98]).

Laplace series (where $\alpha=\beta=0$) come within the case $-1/2<\alpha<1/2$.

However, the more refined results for $\alpha=-1/2$ and $-1/2<\alpha<1/2$ require rather detailed analysis involving careful use of some properties of Bessel functions and are postponed so as to form the content of Part II.

In this part we establish not quite so precise a refinement of (2), but one which is valid over a larger range and for which a different and easier method avails:

$$L_n(\alpha, \beta) = A_{\alpha\beta} n^{\alpha+1/2} + O(n^{\alpha-1/2}) + O(n^{\alpha-\beta-1}) + O(1), \tag{6}$$

for $\alpha>-1/2$, *provided* $\alpha\neq 1/2$, $\beta\neq -1/2$. *If* $\alpha=1/2$, *then* $O(n^{\alpha-1/2})$ *must be replaced by* $O(\log n)$, *and if* $\beta=-1/2$, *then* $O(n^{\alpha-\beta-1})$ *must be replaced by* $O(n^{\alpha-1/2}\log n)$.

In the cases discussed in Part II, (6) will be refined by replacing the O-terms by explicitly determined constants plus O-terms which are $o(1)$.

3. **Proof of** (6). Here $\alpha>-1/2$ and we start with the representation (1), in connection with which we need also

$$\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} = \frac{n^{\alpha+1}}{\Gamma(\alpha+1)} + O(n^{\alpha}) = O(n^{\alpha+1}), \tag{7}$$

an easy consequence of Stirling's formula.

This proof is based essentially on a standard asymptotic expansion of Jacobi polynomials which cannot, unfortunately, be applied throughout the entire interval $(0, \pi)$. Thus, it is convenient to decompose $L_n(\alpha, \beta)$:

$$L_n(\alpha,\beta) = \int_0^{1/n} + \int_{\pi-1/n}^{\pi} + \int_{1/n}^{\pi-1/n} \equiv D_n + E_n + L_n^*(\alpha, \beta), \tag{8}$$

where the same factor (7) which precedes the integral sign in (1) is understood to precede each of the integral signs in (8).

Now, from [5, p. 164, (7.32.5)], since $|\sin\theta|\leqq\theta$ and $|\cos\theta|\leqq 1$,

$$D_n = O(n^{\alpha+1})\int_0^{1/n} O(n^{\alpha+1})\theta^{2\alpha+1}d\theta = O(1).$$

To estimate E_n, we replace θ by $\pi-\theta$ and use [5, p. 58, (4.1.3)],

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \tag{9}$$

so that

$$B_n = O(n^{\alpha+1})\int_0^{1/n} (\cos\{\theta/2\})^{2\alpha+1}(\sin\{\theta/2\})^{2\beta+1}\,|\,P_n^{(\beta,\alpha+1)}(\cos\theta)\,|\,d\theta$$

which, from [5, p. 164, (7.32.5)] again, is

$$O(n^{\alpha+1})\int_0^{1/n}\theta^{2\beta+1}O(n^\beta)d\theta = O(n^{\alpha-\beta-1}).$$

Thus

$$L_n(\alpha,\beta) = L_n^*(\alpha,\beta) + O(n^{\alpha-\beta-1}) + O(1). \tag{10}$$

Applying [5, p. 192, (8.21.18)] to the integrand of $L_n^*(\alpha,\beta)$, we have

$$(\sin\{\theta/2\})^{2\alpha+1}(\cos\{\theta/2\})^{2\beta+1}\,|\,P_n^{(\alpha+1,\beta)}(\cos\theta)\,|$$
$$= (n\pi)^{-1/2}(\sin\{\theta/2\})^{\alpha-1/2}(\cos\{\theta/2\})^{\beta+1/2}\,|\cos(N\theta+\gamma)+(n\sin\theta)^{-1}O(1)\,|$$

where $N=n+(1/2)(\alpha+\beta+2)$ and $\gamma=-(1/2)(\alpha+3/2)\pi$.

We consider first the contribution R_n which the remainder term above makes to $L_n^*(\alpha,\beta)$. This is

$$O(n^{\alpha-1/2})\int_{1/n}^{\pi-1/n}(\sin\{\theta/2\})^{\alpha-3/2}(\cos\{\theta/2\})^{\beta-1/2}d\theta$$
$$= O(n^{\alpha-1/2})\left[\int_{1/n}^{\pi/2}+\int_{\pi/2}^{\pi-1/n}\right].$$

Now,

$$\int_{1/n}^{\pi/2}(\sin\{\theta/2\})^{\alpha-3/2}(\cos\{\theta/2\})^{\beta-1/2}d\theta = \int_{1/n}^{\pi/2}O(\theta^{\alpha-3/2})d\theta$$
$$= \begin{cases} O(n^{1/2-\alpha})+O(1), & \alpha\neq 1/2,\\ O(\log n), & \alpha = 1/2,\end{cases}$$

and

$$\int_{\pi/2}^{\pi-1/n}(\sin\{\theta/2\})^{\alpha-3/2}(\cos\{\theta/2\})^{\beta-1/2}d\theta$$
$$= \int_{1/n}^{\pi/2}(\cos\{\theta/2\})^{\alpha-3/2}(\sin\{\theta/2\})^{\beta-1/2}d\theta = \int_{1/n}^{\pi/2}O(\theta^{\beta-1/2})d\theta$$
$$= \begin{cases} O(n^{-\beta-1/2})+O(1), & \beta\neq -1/2,\\ O(\log n), & \beta = -1/2.\end{cases}$$

Thus, the remainder term does not contribute more to $L_n^*(\alpha,\beta)$, nor, hence, to $L_n(\alpha,\beta)$ than the O-terms in (6).

Consequently,

$$L_n^{\#}(\alpha, \beta) \equiv \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}(n\pi)^{-1/2}$$
$$\cdot \int_{1/n}^{\pi-1/n} (\sin\{\theta/2\})^{\alpha-1/2}(\cos\{\theta/2\})^{\beta+1/2}\,|\cos(N\theta+\gamma)|\,d\theta$$

differs from $L_n(\alpha, \beta)$ only by the O-terms in (6).

Now, $|\cos(N\theta+\gamma)|$ can be replaced in $L_n^{\#}(\alpha, \beta)$ by its mean value, $2/\pi$, with an additive error of $O(n^{\alpha-1/2})+O(n^{\alpha-\beta-2})+O(n^{-1})$. This follows from [2, Theorem 2.1], since the factor of the integral in $L_n^{\#}(\alpha, \beta)$ is $O(n^{\alpha+1/2})$ as stated in (7), provided, of course, that the hypotheses of that theorem are satisfied in the case at hand.

Here the parameter is n instead of x, $a=1/n$, $b=\pi-1/n$, and we may set

$$f_n(\theta) = n^{\alpha+1/2}(\sin\{\theta/2\})^{\alpha-1/2}(\cos\{\theta/2\})^{\beta+1/2}.$$

Hence,

(i) $f_n(\pi-1/n) = n^{\alpha+1/2}(\cos(1/2n))^{\alpha-1/2}(\sin(1/2n))^{\beta+1/2} = O(n^{\alpha-\beta})$

and it remains to establish

(ii) $$\int_{1/n}^{\pi-1/n} |f_n'(\theta)|\,d\theta = O(n^{\alpha+1/2}) + O(n^{\alpha-\beta}) + O(1).$$

This can be verified by straightforward calculations in which it is convenient to treat separately the cases $\beta=-1/2$ and $\beta\neq-1/2$.

Thus, Theorem 2.1 of [2] is applicable, and so

$$L_n^{\#}(\alpha, \beta)$$
$$= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}(n\pi)^{-1/2}\int_{1/n}^{\pi-1/n}(\sin\{\theta/2\})^{\alpha-1/2}(\cos\{\theta/2\})^{\beta+1/2}d\theta$$
$$+ O(n^{\alpha-1/2}) + O(n^{\alpha-\beta-1}) + O(n^{-1}).$$

The limits of integration can be changed to 0 and π with additive errors of $O(n^{\alpha-\beta-1})$ and $O(1)$, respectively, as a consequence of (7). Doing so yields an integral familiar from the theory of the gamma function, namely, a standard form of the beta function.

Finally, using (7) shows that $L_n^{\#}(\alpha, \beta)$ equals the right member of (6). The same is then true of $L_n^{*}(\alpha, \beta)$, which differs from $L_n(\alpha, \beta)$ by the same error terms. Recalling (10) now completes the proof of (6), including the modifications required in the error terms if $\alpha=1/2$ or $\beta=-1/2$.

Remark. Theorem 2.1 of [2], as stated, may appear to cover only integrals which, unlike $L_n^{\#}(\alpha, \beta)$, have constant limits of integration.

Fortunately, however, the integration limits in that theorem can depend on a parameter. This becomes clear on noticing that Lemma 2.1 (loc. cit.), from which the proof of the theorem follows at once on integration by parts, is really nothing more than the obvious fact that the integral of an integrable (in this case, also bounded) periodic function of mean value zero is bounded uniformly in the limits of integration.

This will have to be borne in mind in Part II of this paper as well, where this theorem is used similarly.

4. **A generalization of** (6). (*Added March* 15, 1959). The integral (1) is the special case $\mu=2\alpha+1$, $\lambda=2\beta+1$, of the integral

$$L_n(\alpha,\beta,\mu,\lambda) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}\int_0^\pi (\sin\{\theta/2\})^\mu(\cos\{\theta/2\})^\lambda \cdot \left| P_n^{(\alpha+1,\beta)}(\cos\theta)\right| d\theta.$$

This expression can be studied, as was (1), by the method of §3. Doing so yields an extension of (6) akin to some results of Szegö [**4**; **5**, p. 168, (7.34.1)]:

$$L_n(\alpha,\beta,\mu,\lambda) = A_{\alpha\beta\mu\lambda}n^{\alpha+1/2} + O(n^{\alpha+\beta-\lambda}) + O(n^{2\alpha-\mu+1}) + O(n^{\alpha-1/2}) + O(1),$$

for $\mu>\alpha+1/2$, $\lambda>\beta-1/2$, *provided* $\mu-\alpha\neq 3/2$ *and* $\lambda-\beta\neq 1/2$. *If* $\mu-\alpha=3/2$, *then* $O(n^{2\alpha-\mu+1})+O(n^{\alpha-1/2})$ *must be replaced by* $O(n^{\alpha-1/2}\log n)$. *If* $\lambda-\beta=1/2$, *then* $O(n^{\alpha+\beta-\lambda})+O(n^{\alpha-1/2})$ *must be replaced by* $O(n^{\alpha-1/2}\log n)$. *Here*

$$A_{\alpha\beta\mu\lambda} = \frac{2}{\pi^{3/2}}\,\frac{\Gamma\left(\dfrac{\mu-\alpha-1/2}{2}\right)\Gamma\left(\dfrac{\lambda-\beta+1/2}{2}\right)}{\Gamma(\alpha+1)\Gamma\left(\dfrac{\mu+\lambda-\alpha-\beta}{2}\right)}.$$

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